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Nikolaos Galatos, *University of Denver*, USA
Tommaso Moraschini, *Czech Academy of Sciences*, Czech Republic
Rosalie Iemhoff, *Utrecht University*, The Netherlands

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Preface

This volume contains the extended abstracts of papers presented at the sixth edition of the conference *Logic, Algebra and Truth Degrees* (LATD 2018) held August 28–31, 2018 at the University of Bern, co-located with the twelfth edition of *Advances in Modal Logic* (AiML 2018). Previous editions of the LATD series took place in Siena (2008), Prague (2010), Kanazawa (2012), Vienna (2014), and Phalaborwa (2016).

The plenary talks at LATD 2018 were given by Marta Bílková (Charles University and Czech Academy of Sciences), Nikolaos Galatos (University of Denver), Rosalie Iemhoff (Utrecht University), and Tommaso Moraschini (Czech Academy of Sciences). The program also included 24 contributed talks, selected from 42 submitted papers.

We would like to thank the members of the program committee for their valuable assistance in choosing the invited speakers and conducting the review process. We are also grateful to the members of the organizing committee for their very much appreciated support in hosting the conference: Bettina Choffat, Almudena Colacito, José Gil-Férez, Eveline Lehmann, George Metcalfe (chair), Nenad Savić, Silvia Steila, Thomas Studer (chair), and Olim Tuyt.

Finally, we would like to thank our sponsors for their generous and very welcome support: the Burgergemeinde Bern, Swiss National Science Foundation, University of Bern, Stadt Bern, Association for Symbolic Logic, and European Union Horizon 2020 research and innovation program. We would also like to acknowledge the Easychair conference management system which has greatly facilitated the smooth running of both the submission and review process and compilation of these proceedings.

Bern, August 2018 George Metcalfe

Program Overview

	Monday	Tuesday	Wednesday	Thursday	Friday
9:00	Rob Goldblatt (AiML)	Rosalie Iemhoff (AiML & LATD)	Marta Bílková (LATD)	Agata Ciabattoni (AiML)	Stanislav Kikot (AiML)
10:00	Coffee Break	Coffee Break	Coffee Break	Coffee Break	Coffee Break
10:30	Session 1 (AiML)	Session 3 (AiML & LATD)	Session 5 (AiML & LATD)	Session 6 (AiML & LATD)	Session 8 (AiML & LATD)
12:30	Lunch	Lunch	Free Afternoon	Lunch	Lunch
14:00	Session 2 (AiML)	Tommaso Moraschini (LATD)		Nick Galatos (LATD)	Session 9 (AiML & LATD)
15:00		Coffee Break		Coffee Break	
15:30	Coffee Break	Session 4 (AiML & LATD)		Session 7 (AiML & LATD)	
16:00	Short Papers (AiML)				Farewell Coffee
18:00		Welcome Reception (Schloss Köniz)			
19:00				Social Dinner (Schwellenmätteli Restaurant)	

Contents

Committees **i**

Preface **iii**

Program Overview **v**

Invited Speakers **1**

Logics of information and belief, coalgebraically . . . 3
Marta Bílková

Embedding lattice-ordered bi-monoids in involutive commutative residuated lattices . . . 4
Nikolaos Galatos (joint work with Adam Přenosil)

Regular properties and the existence of proof systems . . . 5
Rosalie Iemhoff

Relational semantics, ordered algebras, and quantifiers for deductive systems . . . 6
Tommaso Moraschini (joint work with Ramon Jansana)

Contributed Papers **11**

First order Gödel logics with propositional quantifiers . . . 13
Matthias Baaz and Norbert Preining

Maximality of First-order logics based on finite MTL-chains . . . 17
Guillermo Badia and Carles Noguera

Partial fuzzy modal logic with a crisp and total accessibility relation . . . 20
Libor Behounek and Antonín Dvořák

MacNeille transferability of finite lattices . . . 24
Guram Bezhanishvili, John Harding, Julia Ilin, and Frederik M. Lauridsen

Epistemic MV-Algebras . . . 27
Manuela Busaniche, Penélope Cordero, and Ricardo Óscar Rodríguez

General neighborhood and Kripke semantics for modal many-valued logics . . . 31
Petr Cintula, Paula Menchón, and Carles Noguera

Hereditarily structurally complete positive logics . . . 35
Alex Citkin

Maximality in finite-valued Łukasiewicz logics defined by order filters 39
Marcelo Coniglio, Francesc Esteva, Joan Gispert, and Lluís Godo

On an implication-free reduct of MV_n chains 43
Marcelo Coniglio, Tommaso Flaminio, Francesc Esteva, and Lluís Godo

A fuzzy-paraconsistent version of basic hybrid logic 47
Diana Costa and Manuel A. Martins

Functionality property in partial fuzzy logic 51
Martina Daňková

Omitting types theorem in mathematical fuzzy logic 55
Denisa Diaconescu and Petr Cintula

Skolemization and Herbrand theorems for lattice-valued Logics 58
Denisa Diaconescu, Petr Cintula, and George Metcalfe

Residuated structures with functional frames 60
Wesley Fussner and Alessandra Palmigiano

$\mathbf{IUL}^{fp}$ enjoys finite strong standard completeness 63
Sándor Jenei

Partially-ordered multi-type algebras, display calculi
and the category of weakening relations 67
Peter Jipsen, Fei Liang, M. Andrew Moshier, and Apostolos Tzimoulis

A many-sorted polyadic modal logic 70
Ioana Leustean and Natalia Moangă

Epimorphisms in varieties of square-increasing residuated structures 74
Tommaso Moraschini, James Raftery, and Jamie Wannenburg

Epimorphisms, definability and cardinalities 77
Tommaso Moraschini, James Raftery, and Jamie Wannenburg

Residuated lattices and the Nelson identity 81
Umberto Rivieccio, Thiago N. Silva, and Matthew Spinks

Modal logics for reasoning about weighted graphs 84
Igor Sedlar and Amanda Vidal Wandelmer

Deciding active structural completeness 87
Michał Stronkowski

Non axiomatizability of the finitary Łukasiewicz modal logic 89
Amanda Vidal

Implicational tonoid logics and their relational semantics 93
Eunsuk Yang and J. Michael Dunn

Invited Speakers

Logics of information and belief, coalgebraically

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To model beliefs of rational agents logically, we switch perspective from a traditional, epistemic alternatives based semantical approach, to information based approach, and see beliefs as based on available information or reasonable expectations. Here the modalities of belief can naturally be seen as diamonds interpreted over information states or probability distributions. In the former case, the corresponding notion of belief is that of confirmed-by-evidence belief. Such epistemic logics have been investigated e.g. as modal extensions of distributive substructural logics [1, 4, 3]. These logical models need to take into account inconsistencies and incompleteness of information, or uncertainty how likely an event is, based on the evidence locally available to agents. This naturally leads us to study in general modal extensions of non-classical logics such as substructural, paraconsistent or many-valued.

In this talk, we will take a general perspective on such modal logics and see them as part of a broader picture of coalgebraic logics. This allows us to address some metamathematical properties of the logics in a convenient level of generality, building on as well as generalizing insights available in coalgebraic logics based on classical logic (detailed references will be given during the talk). As motivating and running examples we shall mostly use modal logics extending Belnap-Dunn four valued logic and Łukasiewicz logic. In particular, we will be interested in completeness and expressivity results.

As understanding the notion of common belief seems to be crucial to a logical account of group beliefs and their dynamics, one of the minor aims of this talk is to present common belief extensions of some epistemic logics based on information states semantics, and prove their completeness. We will consider both finitary and infinitary proof theory of those. The strong completeness of the infinitary versions of the logics requires a proper version of extension lemmata such as Lindenbaum lemma or Belnap's pair-extension lemma. We can offer a general abstract algebraic perspective at both lemmata for infinitary logics, widening the area of their applicability beyond modal extensions of classical logic, and pointing at their limits [2].

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Embedding lattice-ordered bi-monoids in involutive commutative residuated lattices.

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Abstract

We describe a doubly-dense embedding of a lattice-ordered bi-monoid in a commutative residuated lattice. Applications include some old and some new categorical equivalences.

Involutive residuated lattices are important structures and include Boolean algebras, lattice-ordered groups, MV-algebras and Sugihara monoids, among others. They exhibit a pleasing symmetry between the meet and join operations, as well as between multiplication and addition, where the operations in each pair are De Morgan duals of each other, via the negation operation.

We are interested in constructions that produce commutative involutive residuated lattices and in particular where the negation operation is the one that needs to be added, by extending an algebraic structure that has the operations of join, meet, multiplication and addition and satisfies some very natural conditions (most notably the distributivity of multiplication over join and of addition over meet, as well as the *hemidistributivity* axiom $x(y + z) \leq xy + z$); we call such structures *lattice-ordered bi-monoids*.

The main result is an embedding of a lattice-ordered bi-monoid into an involutive commutative residuated lattice such that the original structure sits in a doubly-dense way, reminiscent of the Dedekind MacNeille completion of a lattice. The proof makes use the machinery of residuated frames tailored to the involutive case and the construction of a suitable such involutive frame.

We present applications of the embedding and show that the passing from distributive lattices to Boolean algebras is one example and also the passing from a semilinear Heyting algebra to an odd Sugihara monoid is another example. The addition of the negations is of very different nature in these two examples, as the negation constant is either the bottom or the top of the structure. We are able to obtain new applications, for example by dropping the semilinearity assumption, thus obtaining a categorical equivalence between arbitrary Heyting algebras and non-distributive generalizations of odd Sugihara monoids.

Regular properties and the existence of proof systems

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During the last hundred years proof systems of all kinds have been developed for a great variety of logics. These proof systems can often been used to establish that the corresponding logics have nice properties, such as decidability, interpolation or Skolemization. Results stating that a logic does not have certain proof systems are less common. In this talk a method is introduced to prove such negative results. The method establishes a connection between the existence of certain proof systems for a logic and certain regular properties that the logic satisfies. The talk focusses on (intuitionistic) modal logics, although the method is applicable to other logics as well. The regular properties considered in this talk are variants of interpolation, and the developed method can be used not only to obtain the negative results, but also to prove uniform interpolation for several classical and intuitionistic modal logics. The method is in fact inspired by the syntactic proof that intuitionistic logic has uniform interpolation by Pitts (1992).

The method makes use of sequent calculi, but in a very abstract form. The key property of rules that this method uses is that of being *focussed*, a property that expresses the structurality of a rule. Many of the standard sequent rules for connectives have this property and thus are focussed. In (Iemhoff, 2016) it is shown that if a modal logic has a proof system that consists of focussed rules, then it has uniform interpolation, which implies that the many modal logics without uniform interpolation (Ghilardi and Zawadowski, 2002; Maxsimova, 1977) cannot have focussed proof systems. The generality of the notions involved makes the method applicable to many other logics, for example to intermediate logics.

In how far other proof systems lend themselves to this apporach is still not clear. Besides the technical results above, such unresolved issues as well as related conjectures will be addressed during the talk.

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Relational semantics, ordered algebras, and quantifiers for deductive systems

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Abstract

In this talk we sketch an abstract approach to two-sorted relational semantics based on polarities, whose main advantage is that it applies to arbitrary logics and classes of ordered algebras.

Relational semantics has proved to be a fundamental tool in the investigations of non-classical logics and ordered algebras. Very roughly speaking, logicians may find relational semantics appealing because it allows a *freedom of construction* (i.e. the possibility of deforming models, adding points to them etc.), which is not immediately available in semantics with a more algebraic flavour. On the other hand, algebraists may encounter relational semantics on their way in the form of the skeleton underlying the *theory of completions* of ordered algebras (including Dedekind-MacNeille completions and canonical extensions) and of *Priestley-style dualities*. Finally, philosophers may find relational semantics valuable because of its natural interpretation in terms of *possible worlds* (or states of knowledge, bodies of information etc.), and its relevance in the foundation of *constructive mathematics*.

After Jónsson and Tarski's seminal representation of Boolean algebras with operators and the subsequent discovery of Beth and Kripke semantics for modal and intuitionistic logics, the theory of relational semantics developed in two parallel directions. On the one hand, Bimbó, Dunn and Hardegree among others developed *gaggle theory* as an attempt to provide a general path towards relational semantics for non-classical logics [1, 2, 3, 4, 5]. On the other hand, Gehrke, Harding, Jónsson and others introduced *canonical extensions* to provide an algebraic formulation of topological dualities and a systematic method to produce well-behaved completions of ordered algebras [8, 9, 6, 10].

Remarkably, the theories of gaggles and canonical extensions can be unified under the formalism of completions arising from some special two-sorted relational structures called polarities [7]. More precisely, a *polarity* is a triple $\langle W, J, R\rangle$ where W and J are non-empty sets and $R \subseteq W \times J$. The connection between polarities and completions arises from the observation that every polarity $\langle W, J, R\rangle$ induces a complete lattice. To explain why, consider the Galois connection $(\cdot)^{\rhd} \colon \mathcal{P}(W) \longleftrightarrow \mathcal{P}(J) \colon (\cdot)^{\lhd}$ defined for every $A \subseteq W$ and $B \subseteq J$ as

$$A^{\rhd} \coloneqq \{j \in J : A \times \{j\} \subseteq R\} \text{ and } B^{\lhd} \coloneqq \{w \in W : \{w\} \times B \subseteq R\}.$$

From the general theory of Galois connections, it follows that the map $(\cdot)^{\rhd\lhd}$ is a closure operator on W. We denote by $\mathcal{G}(W, J, R)$ the complete lattice of closed sets of $(\cdot)^{\rhd\lhd}$ induced by the polarity $\langle W, J, R\rangle$. Moreover, we denote by $\leqslant_W$ and $\leqslant_J$ the preorders on W and J respectively defined by the following conditions:

$$w \leqslant_W u \iff u^{\rhd\lhd} \subseteq w^{\rhd\lhd} \quad \text{and} \quad j \leqslant_J i \iff j^{\lhd} \subseteq i^{\lhd}.$$

In this talk we sketch a first abstract approach to two-sorted relational semantics based on polarities, whose main advantage is that it applies to arbitrary logics and classes of ordered algebras (regardless of the presence of a lattice reduct). The starting point of our discussion is a basic question, which needs to be addressed by any truly general theory of relational semantics:

what does it mean that a logic has a relational semantics?

The fact that such a fundamental question has been somehow overlooked by the literature should be attributed to the fact that in most cases our belief that some logics (such as modal and intuitionistic ones) have a relational semantics is largely motivated on empiric observations rather than on conceptual clarification. Whatever the answer to the above question may be, it is clear that, in order to determine whether a logic has a relational semantics, we should first clarify what is a relational structure in general.

General frames and ordered algebras

An *ordered language* is an algebraic language $\mathscr{L}$, equipped with an assignment to every basic operation symbol $f \in \mathscr{L}$ of a choice of which arguments of f will be treated as increasing and which ones as decreasing. In this case, given $f \in \mathscr{L}$, we write

$$f = f(\vec{x}; \vec{y}) = f(x_1, \ldots, x_m; y_1, \ldots, y_n)$$

to denote the fact that the variables $\vec{x}$ will be treated as increasing and the variables $\vec{y}$ as decreasing.

Given an ordered language $\mathscr{L}$, an *$\mathscr{L}$-algebra* is a pair $\langle \boldsymbol{A}, \leqslant \rangle$ where $\boldsymbol{A}$ is an algebra and $\langle A, \leqslant \rangle$ is a poset such that $f^{\boldsymbol{A}}(\vec{x}; \vec{y})$ is increasing on $\vec{x}$ and decreasing on $\vec{y}$ with respect to $\leqslant$, for every $f(\vec{x}; \vec{y}) \in \mathscr{L}$. For instance, when ordered under the lattice order, Heyting algebras, modal algebras, and residuated lattices are $\mathscr{L}$-algebras for natural ordered languages $\mathscr{L}$.

Dealing with relational semantics it is convenient to think of all basic operations as generalizations of one of the two basic modal operations $\Box$ and $\Diamond$. Accordingly, we say that a *labeling map* for an ordered language $\mathscr{L}$ is a function $\beta \colon \mathscr{L} \to \{\Diamond, \Box\}$. A *labeled ordered language $\mathscr{L}$* is an ordered language $\mathscr{L}$ equipped with a labeling map β.

Given a labeled ordered language $\mathscr{L}$, an *$\mathscr{L}$-frame* is a structure

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\} \rangle$$

where $\langle W, J, R \rangle$ is a polarity s.t. $\leqslant_W$ and $\leqslant_J$ are partial orders, and for every operation symbol $f \in \mathscr{L}$ s.t. $f = f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ and $\beta(f) = \Diamond$, we have that $T_f \subseteq W^m \times J^n \times W$ and

1. For all $\vec{w}_1, \vec{w}_2 \in W^m$, $\vec{j}_1, \vec{j}_2 \in J^n$, and $u_1, u_2 \in W$ s.t. $\vec{w}_2 \leqslant_W \vec{w}_1$, $\vec{j}_2 \leqslant_J \vec{j}_1$ and $u_1 \leqslant_W u_2$,
$$\text{if } \langle \vec{w}_1, \vec{j}_1, u_1 \rangle \in T_f, \text{ then } \langle \vec{w}_2, \vec{j}_2, u_2 \rangle \in T_f.$$
2. $\{u \in W : \langle \vec{w}, \vec{j}, u \rangle \in T_f\}$ is a closed set of $(\cdot)^{\rhd\lhd}$ for all $\vec{w} \in W^m$ and $\vec{j} \in J^n$.

If $\beta(f) = \Box$, a dual requirement is asked. We refer to W and J as to the sets of *worlds* and *co-worlds* of $\boldsymbol{F}$.

As in the case of modal and intuitionistic logics, every $\mathscr{L}$-frame $\boldsymbol{F}$ can be associated with a complete $\mathscr{L}$-algebra $\boldsymbol{F}^+$ whose universe consists in a collection of distinguished sets of worlds, namely $\mathcal{G}(W, J, R)$. Accordingly, an *$\mathscr{L}$-general frame* is a pair $\langle \boldsymbol{F}, A \rangle$ where $\boldsymbol{F}$ is an $\mathscr{L}$-frame and A is the universe of a substructure of $\boldsymbol{F}^+$. The *complex algebra* $\langle \boldsymbol{F}, A \rangle^+$ of $\langle \boldsymbol{F}, A \rangle$ is the substructure of $\boldsymbol{F}^+$ with universe A, which is of course an $\mathscr{L}$-algebra.

Local consequences

A valuation in a general frame $\langle \boldsymbol{F}, A\rangle$ is a map $v\colon Var \to A$, where Var is fixed countable set of variables. It is possible to define a notion of *satisfaction* at a world w and a notion of *co-satisfaction* at a co-world j of a formula φ (in variables Var) under a valuation v, in symbols $w, v \Vdash \varphi$ and $j, v \succ \varphi$. This allows to associate two consequence relations to every class Fr of $\mathscr{L}$-general frames:

1. The *local consequence* of Fr, in symbols $\vdash^{l}_{\mathsf{Fr}}$, is defined as follows:

$$\begin{aligned}\Gamma \vdash^{l}_{\mathsf{Fr}} \varphi \Longleftrightarrow\ & \text{for all } \langle \boldsymbol{F}, A\rangle \in \mathsf{Fr}, \text{ valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle, \text{ and } w \in W,\\ & \text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.\end{aligned}$$

2. The *co-local consequence* of Fr, in symbols $\vdash^{cl}_{\mathsf{Fr}}$, is defined as follows:

$$\begin{aligned}\Gamma \vdash^{cl}_{\mathsf{Fr}} \varphi \Longleftrightarrow\ & \text{for all } \langle \boldsymbol{F}, A\rangle \in \mathsf{Fr}, \text{ valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle, \text{ and } j \in J,\\ & \text{if } j, v \succ \Gamma, \text{ then } j, v \succ \varphi.\end{aligned}$$

Motivated by the above observation, we say that a logic $\vdash$ is an *$\mathscr{L}$-local consequence* if it is the local consequence of some class of $\mathscr{L}$-general frames. A closer look at the theory of $\mathscr{L}$-local consequences shows that each of them can be associated with a distinguished class of $\mathscr{L}$-algebras and of $\mathscr{L}$-general frames, which are related by a sort of weak duality (one half of which is given by the complex-algebra construction). As a matter of fact, this approach encompasses that of canonical extensions of arbitrary lattices, but diverges from it when applied to non-lattice based logics. This is due to the fact that our approach produces logic-based completions, i.e. completions which reflect to some extent the behaviour of the logic under consideration. This makes it especially fruitful in the study of purely intensional fragments of substructural logics (which are not lattice-based).

If time allows, we will discuss the fact that any local consequences can be semantically expanded to the first-order level with identity, universal and existential quantifiers. Remarkably, this expansion is almost always conservative with respect to the original propositional logic, and can be axiomatized by means of a transparent sequent calculus. Moreover, this construction yields a sound and complete semantics for all (local consequences of) modal and superintuitinistic first-order logics.

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Contributed Papers

First Order Gödel Logics with Propositional Quantifiers

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Abstract

We extend first-order Order Gödel Logics by propositional quantifiers.

1 Introduction

First order Gödel logics form a well established class of many-valued logics with good proof-theoretic properties and extensive theory [4].

Quantified Propositional Gödel Logics have been studied only a few times, and not much is known about these logics besides a few results for specific logics [1–3, 6]. Propositional quantifiers allow for quantification of propositions, which in the setting of Gödel logics boils down to quantification over all truth values of the underlying truth-value set. Thus, they are somewhere between first order and second order quantifiers.

It has already been shown that there are uncountably many quantified propositional Gödel logics (without first order quantifiers), which is in stark contrast to the fact that there are only countably many first order Gödel logics [7].

For three cases, namely $V_\infty = [0, 1]$, $V_\downarrow = \{0\} \cup \{1/n : n > 0\}$, and $V_\uparrow = \{1\} \cup \{1 - 1/n : n > 0\}$, quantifier elimination has been shown for the propositional quantified Gödel logics, in later two cases by extension of the language with an additional operator [1, 3].

In this paper we initiate the research program to study the combination of propositional and first-order quantifieres with respect to Gödel logics.

1.1 Syntax and semantics for Gödel logics

For the following let us fix a countably infinite set $\mathscr{P}$ of propositional variables, usually written as p, q, ..., and a *countable* first-order language $\mathscr{L}$ which also contains $\mathscr{P}$.

The definition of formulas in the Gödel logics with propositional quantifiers is as with standard first-order Gödel logics (see [4]) with the additional clause that propositional variables from $\mathscr{P}$ also count as atomic formulae, and that we allow quantification over propositional variables: $\forall^p q A(q)$ and $\exists^p q A(q)$.

The semantics of first-order Gödel logics with propositional quantifiers, with respect to a fixed Gödel set as truth value set and $\mathscr{L}$ is defined using the extended language $\mathscr{L}^{M,V}$, where M is a universe of objects. $\mathscr{L}^{M,V}$ is $\mathscr{L}$ extended with symbols for every element of M as constants, so called M-symbols, as well as constants r for each $r \in V$, the underlying truth value set. These symbols are denoted with the same letters.

Definition 1 (Semantics of Gödel logics with propositional quantifiers)**.** Fix a Gödel set V. A valuation v into V consists of (1) a nonempty set $M = M^v$, the 'universe' of v, and (2) for each k-ary predicate symbol P, a function $P^v : M^k \to V$.

Given a valuation v, we can naturally define a value $v(A)$ for any closed formula A of $\mathscr{L}^M$. For atomic formulas $A = P(m_1, \dots, m_n)$, we define $v(A) = P^v(m_1, \dots, m_n)$. For atomic formulas $A = r$, we define $v(A) = r$. For composite formulas A we define $v(A)$ naturally by: $v(\bot) = 0$, $v(A \wedge B) = \min\{v(A), v(B)\}$, $v(A \vee B) = \max\{v(A), v(B)\}$, $v(A \to B) = v(B)$ if $v(A) > v(B)$, otherwise $= 1$, and the quantifier rules

$$v(\forall x A(x)) = \inf\{v(A(m)) : m \in M\} \qquad v(\exists x A(x)) = \sup\{v(A(m)) : m \in M\}$$
$$v(\forall^p q A(q)) = \inf\{v(A(r)) : r \in V\} \qquad v(\exists^p q A(q)) = \sup\{v(A(r)) : r \in V\}$$

(Here we use the fact that our Gödel sets V are *closed* subsets of $[0, 1]$, in order to be able to interpret $\forall$ and $\exists$ as inf and sup in V.)

For any closed formula $A \in \mathscr{L}$ and any Gödel set V we let

$$\|A\|_V := \inf\{v(A) : v \text{ a valuation into } V\}$$

Definition 2 (Gödel logic with propositional quantifiers based on V)**.** Let V be a Gödel set. The *first order Gödel logic with propositional quantifiers* $\mathbf{G}^{\mathbf{foqp}}_{\mathbf{V}}$*,* as the set of all closed formulas of $\mathscr{L}$, such that $\|A\|_V = 1$. The set $\mathbf{G}^{\mathbf{foqp}}_{\mathbf{V}}$ is also written as $\mathbf{VAL}^{\mathbf{foqp}}_{\mathbf{V}}$.

In the following we will refer to $\mathbf{G}^{\mathbf{foqp}}_{[\mathbf{0},\mathbf{1}]}$ simply as $\mathbf{G}^{\mathbf{foqp}}$

2 Properties of $\mathbf{G}^{\mathbf{foqp}}$ and $\bigcap_V \mathbf{G}^{\mathbf{foqp}}_{\mathbf{V}}$

Lemma 3. *Both directions of the density axiom*

$$\forall^p p((A \to p) \vee (p \to B)) \leftrightarrow (A \to B)$$

hold in $\mathbf{G}^{\mathbf{foqp}}$.

Proof. First assume that $v(A) \le v(B)$, then the valuation of the right side is $= 1$, and whatever $v(p)$ is evaluated to, it is either $\ge v(A)$ or $\le v(B)$, and thus the valuation of the left side is also $= 1$.

In the case that $v(A) > v(B)$, the valuation of the right side is $= v(B)$. We compute the value of $(A \to p \vee p \to B)$ by looking at the possible valuation of p: (i) $v(p) \le v(B)$: the valuation is $= 1$; (ii) $v(p) \ge v(A)$: the valuation is again $= 1$; (iii) $v(B) < p < v(A)$: the valuation is $= v(p)$. Since we are working in $\mathbf{G}^{\mathbf{foqp}}$, there are real numbers strictly between $v(B)$ and $v(A)$. Thus, we obtain $v(\forall^p p(A \to p \vee p \to B)) = \inf_{v(B)<p<v(A)} v(p) = v(B)$, which concludes the proof. □

Theorem 4. *There is no Gödel set V such that* $\mathbf{G}^{\mathbf{foqp}}_{\mathbf{V}} = \bigcap_W \mathbf{G}^{\mathbf{foqp}}_{\mathbf{W}}$.

Proof. If 0 is isolated in V, then $\exists u \forall w(\neg\neg u \wedge (\neg w \vee u \to w))$ is valid in $\mathbf{G}^{\mathbf{foqp}}_{\mathbf{V}}$. If 0 is not isolated in V, then $\neg\forall w \neg\neg w$ is valid in $\mathbf{G}^{\mathbf{foqp}}_{\mathbf{V}}$. But none of the two is valid in $\bigcap_W \mathbf{G}^{\mathbf{foqp}}_{\mathbf{W}}$. □

Corollary 5. $\mathbf{G}^{\mathbf{foqp}} \ne \bigcap_W \mathbf{G}^{\mathbf{foqp}}_{\mathbf{W}}$

Theorem 6. *Every formula in* $\mathbf{G}^{\mathbf{foqp}}$ *has an equivalent prenex normal form.*

Proof. The two propositional and first order quantifier shifts that are not generally valid in Gödel logics are

$$(\forall x A(x) \to B) \to \exists x(A(x) \to B) \tag{i}$$
$$(B \to \exists x A(x)) \to \exists x(B \to A(x)) \tag{ii}$$
$$(\forall^p p A(p) \to B) \to \exists^p p(A(p) \to B) \tag{iii}$$
$$(B \to \exists^p p A(p)) \to \exists^p p(B \to A(p)) \tag{iv}$$

(see [4], Section 3.2). Let us first look at (i): We show that

$$(\forall x A(x) \to B) \to \exists x((A(x) \to p) \vee (p \to B)) \tag{*}$$

holds in $\mathbf{G}^{\mathbf{foqp}}$: If $v(\forall x A(x)) > v(B)$, the valuation of the left side $= v(B)$. On the other hand it is obvious that the valuation of the right side is always $\geq v(B)$, and as a consequence the valuation of (*) is $= 1$.

If $v(\forall x A(x)) \leq v(B)$, the valuation of the left side $= 1$, and we need to show that the valuation of the right side is also $= 1$. We compute the value of the right side for all possible valuations of p:

(a) $v(p) \leq v(B)$: In this case the existential quantifier over x is irrelevant since the second part of the disjunction $p \to B$ always evaluates to $= 1$, and consequently also the whole right side evaluates to $= 1$.

(b) $v(B) < v(p)$: Since we also have that $v(\forall x A(x)) \leq v(B)$, we have $v(\forall x A(x)) \leq v(B) < v(p)$. That means that for some c of the object universe, $v(A(c)) < v(p)$, and thus $v((A(c) \to p) \vee (p \to B)) = 1$. We obtain that also the valuation of the right side $= 1$.

Having shown that (*) holds for all p, we obtain that the following implication also holds:

$$(\forall x A(x) \to B) \to \forall^p p \exists x((A(x) \to p) \vee (p \to B))$$

which provides the necessary quantifier shift for the case (i).

The case of (ii)-(iv) can be treated in a similar way. □

Theorem 7. *Every formula in* $\mathbf{G}^{\mathbf{foqp}}$ *without quantified propositional quantifiers has a normal form of the following form*

$$\exists^p \vec{p} \exists \vec{x} \forall \vec{y} A'(\vec{p}, \vec{x}, \vec{y}).$$

Proof. Write A in structural normal form. This structural normal for can be written as

$$\bigwedge \forall \bar{x}_i A_i(\bar{x}_i) \wedge \bigwedge \forall \bar{y}_j (B_j(\bar{y}_j) \to \exists^p a B'_j(a, \bar{y}_j)) \wedge \bigwedge \forall \bar{z}_k (\forall^p b C'_k(b, \bar{z}_k) \to C_k(\bar{z}_k)) \to F$$

where A_i, B_j, B'_j, C_k, C'_k, and F are quantifier free. As in the proof of Theorem 6 replace $\forall \bar{y}_j (B_j(\bar{y}_j) \to \exists^p a B'_j(a, \bar{y}_j))$ by $\forall \bar{y}_j \forall^p p \exists^p a(B_j(\bar{y}_j) \to B(a, \bar{y}_j))$, and $\forall \bar{z}_k (\forall^p b C'_k(b, \bar{z}_k) \to C_k(\bar{z}_k))$ by $\forall \bar{z}_k \forall^p p \exists b(C'_k(b, \bar{z}_k) \to C_k(\bar{z}_k))$.

Note that $\forall \bar{x} A(x) \to B$ is valid in $\mathbf{G}^{\mathbf{foqp}}$ is equivalent to $\exists \bar{x}(A(\bar{x}) \to C \vee C \to B)$ valid in $\mathbf{G}^{\mathbf{foqp}}$, where C is a new propositional constant and $(\exists \bar{x} A(x) \to B) \to \forall \bar{x}(A(\bar{x}) \to B)$ is valid in all Gödel logics.

□

Theorem 8. *The intersection of all first order Gödel logics with quantifiers is not recursively enumerable.*

Proof. We use the proof method introduced in [5] to show non-re of countable first-order Gödel logics. By defining a exactly countable set of points the assumption of being re would lead to a reduction of classical validity in all finite models to the validity of a formula in the given logic, which by Trakhtenbrot's Theorem is not recursively enumerable.

We refer the reader to [5] for details of the process, and only give the necessary definitions for the exactly countable set: We first use a monadic predicate symbol $P(x)$ to force a descent to 0:

$$\neg P(x) \wedge \forall x \neg\neg P(x).$$

Then we state that for each $P(x)$ there is a surrounding of it that only contains the truth value of $P(x)$ itself:

$$\forall x \exists^p p \exists^p q (p \prec P(x) \prec q \wedge \forall r (p \prec r \prec q \to (P(x) \leftrightarrow q))).$$

Note that these definitions can only be carried out using propositional quantifiers. By this, we obtain at least countably many open intervals, which can be made non-overlapping by symmetric differences (the p and q could coincide with some $P(x)$, which would make half of the intervals overlapping). Since one cannot embed more than countably many open intervals into $[0, 1]$, we obtain exactly countably many open intervals. By relativizing the quantifiers in the original formula to the countably many points we obtain the necessary embedding. □

We finish with the most important open problem, namely whether $\mathbf{G}^{\mathbf{foqp}}$ is recursively enumerable. The obvious axiomatization would be the standard axiomatization of first order Gödel logics together with the default axioms and rules for quantified propositional quantifiers.

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Maximality of First-order Logics Based on Finite MTL-chains

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First-order logic is the best known example of a formal language whose model theory had a great impact on 20th century mathematics (from non-standard analysis to abstract algebra). The celebrated characterization of classical first-order logic obtained by Per Lindström in the 60s (published as [6]) is a landmark in contemporary logic. The introduction of a notion of "extended first-order logic", that encompassed a great number of expressive extensions of first-order logic, allowed Lindström to establish, roughly, that there are no extensions of classical first-order logic (in terms of expressive power) that would also satisfy the compactness and Löwenheim–Skolem theorems (a nice accessible exposition can be found in [4]). Expressive extensions of first-order logic are commonly called "abstract logics".[1]

Lindström's theorem single-handedly started a new area of research known as *abstract* or *soft* model theory (cf. [1, 2]). The terminology comes from the fact that, when working in this field, one uses "only very general properties of the logic, properties that carry over to a large number of other logics" ([1], p. 225). Some common examples of such properties are compactness, the Craig interpolation theorem or the Beth definability theorem. Abstract model theory is concerned with the study of such properties and their mutual interaction.

In the framework of Mathematical Fuzzy Logic (MFL) the possibility of abstract model-theoretic results was briefly considered by Petr Hájek's in a technical report [5]. Indeed, he showed that the analogues of Lindström theorem fails for some of the main first-order fuzzy logics (BL∀, Ł∀, Π∀, and G∀) with their standard semantics. Perhaps discouraged by this initial negative result, the MFL community has not attempted again, to the best of our knowledge, to build a corresponding abstract model theory. In this talk we would like to show that such a theory could actually be a viable one, at least under certain technical conditions.

The algebraic framework is going to be that of MTL-algebras, that is, algebraic structures of the form $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ such that

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a bounded lattice,
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a commutative monoid,
- for each $a, b, c \in A$, we have:

$$a \,\&^{\boldsymbol{A}}\, b \leq c \quad \text{iff} \quad b \leq a \to^{\boldsymbol{A}} c, \qquad \text{(residuation)}$$

$$(a \to^{\boldsymbol{A}} b) \vee^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} a) = \overline{1}^{\boldsymbol{A}} \qquad \text{(prelinearity)}$$

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[1] In fact, probably the term "model-theoretic language" ([3]) is more accurate, depending on one's views of what a "logic" is.

$\boldsymbol{A}$ is called an MTL-*chain* if its underlying lattice is linearly ordered. Henceforth, we will be working with a fixed *finite* MTL-chain $\boldsymbol{A}$.

Our propositional language will have a set of connectives $\mathcal{L} = \{\wedge, \vee, \rightarrow, \&\}$ plus, for any element a of $\boldsymbol{A}$, we will have a constant $\overline{a}$ to denote it. When $a = \overline{1}^{\boldsymbol{A}}$ or $a = \overline{0}^{\boldsymbol{A}}$, then $\overline{a} = \overline{1}$ or $\overline{a} = \overline{0}$, respectively. A predicate language $\mathcal{P}^{\boldsymbol{A}}$ based on a signature τ (which contains relation, function and constant symbols) will have again the basic set of connectives $\mathcal{L} \cup \{\overline{a} \mid a \in A\}$ plus the quantifiers $\exists, \forall$ and a crisp equality $=$.

We say that a sequence $\bar{a}$ of objects from M in a model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ *satisfies* a formula $\varphi(\bar{x})$ if $\|\varphi\|^{\mathfrak{M}}_{v[\bar{x}\rightarrow\bar{a}]} = \overline{1}^{\boldsymbol{A}}$ (also written $\|\varphi[\bar{a}]\|^{\mathfrak{M}} = \overline{1}^{\boldsymbol{A}}$ or as $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \varphi[\bar{a}]$). For a set of formulas Φ, we write $\|\Phi\|^{\mathfrak{M}}_{v} = \overline{1}^{\boldsymbol{A}}$ if $\|\varphi\|^{\mathfrak{M}}_{v} = \overline{1}^{\boldsymbol{A}}$ for every $\varphi \in \Phi$. We say that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a *model of a set of formulas* $\Phi(\bar{x})$ if $\|\varphi[\bar{a}]\|^{\mathfrak{M}} = \overline{1}^{\boldsymbol{A}}$ for each $\varphi(\bar{x}) \in \Phi(\bar{x})$ and some sequence $\bar{a}$ of elements from M.

We will say that two formulas $\varphi(x)$ and $\psi(x)$ are *1-equivalent* if for any model $\mathfrak{M}$ and element of the model e, $\mathfrak{M} \models \varphi[e]$ iff $\mathfrak{M} \models \psi[e]$.

An *abstract logic* will be any pair of the form $\mathscr{L}^{\boldsymbol{A}} = \langle \mathcal{L}_{\mathscr{L}}, \|\|_{\mathscr{L}}\rangle$ such that:

- $\mathcal{L}_{\mathscr{L}}$ maps every signature τ to a set of $\mathscr{L}(\tau)$-sentences $\mathcal{L}_{\mathscr{L}}(\tau)$ such that:
  - If $\tau \subseteq \tau'$, then $\mathcal{L}_{\mathscr{L}}(\tau) \subseteq \mathcal{L}_{\mathscr{L}}(\tau')$.
  - (Occurrence). If $\varphi \in \mathcal{L}_{\mathscr{L}}(\tau)$, then there is a finite $\tau_{\varphi} \subseteq \tau$ such that for every signature τ', $\varphi \in \mathcal{L}_{\mathscr{L}}(\tau')$ iff $\tau_{\varphi} \subseteq \tau'$.
  - (Closure). Every $\mathcal{L}_{\mathscr{L}}(\tau)$ contains τ as a subset and it is closed for the connectives in $\mathcal{L}$.
- $\|\|_{\mathscr{L}}$ is a function which maps every pair $\langle \varphi, \langle \mathfrak{M}, e\rangle\rangle$ to an element of $\boldsymbol{A}$, where, for some signature τ, $\varphi \in \mathcal{L}_{\mathscr{L}}(\tau)$ and $\langle \mathfrak{M}, w\rangle$ is a pair of an $(\boldsymbol{A}, \tau)$-model and element of such model. This function is assumed to respect the interpretation of the basic connectives in $\mathcal{L}$ and the first-order quantifiers,

  in addition to the following conditions:
  - (Isomorphism). Whenever $\varphi \in \mathcal{L}_{\mathscr{L}}(\tau)$, $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{N}, v\rangle$ are $(\boldsymbol{A}, \tau)$-models, and f is an isomorphism between $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{N}, v\rangle$, then
  $$\|\varphi\|^{\langle \mathfrak{N}, v\rangle}_{\mathscr{L}} = \|\varphi\|^{\langle \mathfrak{M}, w\rangle}_{\mathscr{L}} .$$
  - (Expansion). If τ, τ' are two signatures such that $\tau \subseteq \tau'$ and $\varphi \in \mathcal{L}_{\mathscr{L}}(\tau)$, $\langle \mathfrak{M}, w\rangle$ is an $(\boldsymbol{A}, \tau')$-model and $\mathfrak{M} \restriction \tau$ is the reduct of $\mathfrak{M}$ to τ, then
  $$\|\varphi\|^{\langle \mathfrak{M}, w\rangle}_{\mathscr{L}} = \|\varphi\|^{\langle \mathfrak{M}\restriction\tau, w\rangle}_{\mathscr{L}} .$$

We will write $\mathscr{L}_1^{\boldsymbol{A}} \trianglelefteq \mathscr{L}_2^{\boldsymbol{A}}$ to mean that for every formula of the first logic there is a 1-equivalent formula in the second. If both $\mathscr{L}_1^{\boldsymbol{A}} \trianglelefteq \mathscr{L}_2^{\boldsymbol{A}}$ and $\mathscr{L}_2^{\boldsymbol{A}} \trianglelefteq \mathscr{L}_1^{\boldsymbol{A}}$ hold, then we say that the abstract extensions $\mathscr{L}_1^{\boldsymbol{A}}$ and $\mathscr{L}_2^{\boldsymbol{A}}$ are *expressively equivalent* and write $\mathscr{L}_1^{\boldsymbol{A}} \simeq \mathscr{L}_2^{\boldsymbol{A}}$.

Note that $\mathscr{L}_0^{\boldsymbol{A}}$ is obviously among its own abstract extensions, so that these extensions extend $\mathscr{L}_0^{\boldsymbol{A}}$ in the sense of the partial $\trianglelefteq$-order.

We use $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ to denote the abstract logic obtained from considering our first-order languages with constants. The subindexes represent the finitary character of the quantifiers and the connectives $\vee, \wedge, \&$.

By a *crisp predicate* we mean one taking only values in the set $\{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$. In the presence of our crisp equality, a function symbol f can be represented as a crisp binary predicate which is functional.

We will show the proofs of the following main results:

Theorem 1. (First Lindström Theorem) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an abstract logic such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \trianglelefteq \mathscr{L}^{\boldsymbol{A}}$. If $\mathscr{L}^{\boldsymbol{A}}$ has the Löwenheim–Skolem property and the Compactness property for countable sets of formulas, then $\mathscr{L}^{\boldsymbol{A}} \trianglelefteq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.*

Lemma 2. (Separation Lemma) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an abstract logic with the finite occurrence property such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \leq \mathscr{L}^{\boldsymbol{A}}$. If $\mathscr{L}^{\boldsymbol{A}}$ has the Löwenheim-Skolem property and for some τ_0 there are disjoint classes $Mod(\varphi), Mod(\chi)$ (for φ, χ formulas in τ_0 of $\mathscr{L}^{\boldsymbol{A}}$) such that there is no $Mod(\psi)$ (ψ a formula in τ_0 of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$) separating $Mod(\varphi)$ and $Mod(\chi)$, i.e.,*

$$Mod(\varphi) \subseteq Mod(\psi) \quad and \quad Mod(\chi) \cap Mod(\psi) = \emptyset.$$

Then for some signature δ containing at least a unary predicate U there is a formula θ in δ of $\mathscr{L}^{\boldsymbol{A}}$ such that:

(i) $\mathfrak{M} \models \theta$ then $U^{\mathfrak{M}}$ is crisp and finite with cardinality $\geqslant 1$.

(ii) for every $n \geqslant 1$, we can find $\mathfrak{M} \models \theta$ and $|\{a \in M \mid \mathfrak{M} \models U[a]\}| = n$.

Theorem 3. (Second Lindström Theorem) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an effective abstract logic (i.e., the collection of its formulas is recursive) such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \trianglelefteq \mathscr{L}^{\boldsymbol{A}}$. If $\mathscr{L}^{\boldsymbol{A}}$ has the Löwenheim–Skolem property and the abstract Completeness property (the collection of its validities is recursively enumerable), then $\mathscr{L}^{\boldsymbol{A}} \trianglelefteq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.*

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Partial Fuzzy Modal Logic with a Crisp and Total Accessibility Relation*

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1 Introduction

In recent decades, modal logics have been studied not just in the classical bivalent setting, but also in non-classical settings including that of fuzzy logic (e.g., [4, §8.3], [3]). Here we make an initial exploration of modalities over *partial* fuzzy logic, i.e., a variant of fuzzy logic that admits truth-valueless propositions. A simple system of partial fuzzy logic, representing truth-value gaps by an additional truth value added to the algebra of degrees, was proposed recently in [2, 1]. We employ its apparatus to define the semantics of Kripke-style fuzzy modalities that admit truth-value gaps, and investigate their basic properties. While in general not only modal propositions, but also the accessibility relation can be both partially defined and fuzzy, here we focus on the simpler case where the accessibility relation is total and crisp. Partial fuzzy modalities based on partial or fuzzy accessibility relations are left for future work.

2 Partial fuzzy logic

Partial fuzzy logic L^* proposed in [2] represents truth-value gaps by an additional truth value $*$, added to an algebra of truth degrees of an underlying $\triangle$-core [5] fuzzy logic L. The intended (general, linear, standard) L^*-algebras are thus defined as expansions $\boldsymbol{L}_* = \boldsymbol{L} \cup \{*\}$ of (arbitrary, linear, standard) L-algebras $\boldsymbol{L}$, where $* \notin \boldsymbol{L}$. The connectives of $\boldsymbol{L}$ are extended to $\boldsymbol{L}_*$ in several parallel ways, including the following prominent families of $\boldsymbol{L}_*$-connectives:

- The *Bochvar-style* connectives c_{B} for each connective c of L, which treat $*$ as the absorbing element, are defined by the following truth functions on $\boldsymbol{L}_*$, for each $\alpha, \beta \in \boldsymbol{L}$, any unary connective u_{B}, and any binary connective c_{B} (and similarly for higher arities):

	u_{B}
α	$u\alpha$
$*$	$*$

c_{B}	β	$*$
α	$\alpha\, c\, \beta$	$*$
$*$	$*$	$*$

- The *Sobociński-style* connectives $c_{\mathrm{S}} \in \{\wedge_{\mathrm{S}}, \vee_{\mathrm{S}}, \&_{\mathrm{S}}\}$, which treat $*$ as the neutral element, and the Sobociński-style implication $\rightarrow_{\mathrm{S}}$ associated with $\&_{\mathrm{S}}$ by the residuation identity $x \rightarrow_{\mathrm{S}} (y \rightarrow_{\mathrm{S}} z) = (x \,\&_{\mathrm{S}}\, y) \rightarrow_{\mathrm{S}} z$, are defined as follows:

c_{S}	β	$*$
α	$\alpha\, c\, \beta$	α
$*$	β	$*$

$\rightarrow_{\mathrm{S}}$	β	$*$
α	$\alpha \rightarrow \beta$	$\neg\alpha$
$*$	β	$*$

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- The *Kleene-style* connectives $c_{\mathrm{K}} \in \{\wedge_{\mathrm{K}}, \vee_{\mathrm{K}}, \&_{\mathrm{K}}, \rightarrow_{\mathrm{K}}\}$, which preserve the neutral and absorbing elements of the corresponding connectives of L and otherwise are evaluated Bochvar-style, are defined as follows, for all $\alpha, \beta \in \boldsymbol{L} \setminus \{1\}$, $\gamma, \delta \in \boldsymbol{L} \setminus \{0\}$:

$\wedge_{\mathrm{K}}$	0	β	$*$
0	0	0	0
α	0	$\alpha \wedge \beta$	$*$
$*$	0	$*$	$*$

$\vee_{\mathrm{K}}$	δ	1	$*$
γ	$\gamma \vee \delta$	1	$*$
1	1	1	1
$*$	$*$	1	$*$

$\&_{\mathrm{K}}$	0	β	$*$
0	0	0	0
α	0	$\alpha \mathbin{\&} \beta$	$*$
$*$	0	$*$	$*$

$\rightarrow_{\mathrm{K}}$	δ	1	$*$
0	1	1	1
α	$\alpha \rightarrow \delta$	1	$*$
$*$	$*$	1	$*$

- Moreover, several useful auxiliary connectives are available in L^*, including the following unary ones (where $\alpha \in \boldsymbol{L}$):

	$!$	$?$	$\downarrow$	$\uparrow$
α	1	0	α	α
$*$	0	1	0	1

  Further families of L^*-definable connectives are left aside here. Of the listed connectives, only a few need be primitive (e.g., $\{c_{\mathrm{B}} \mid c \in \mathrm{L}\} \cup \{*, !, \wedge_{\mathrm{K}}\}$; see [2, Th. 2.4]).

Based on the intended algebraic semantics described above, the consequence relation of the logic L^* is defined in the standard manner, with the only designated truth value 1. The logic L^* turns out to be Rasiowa-implicative [6] (w.r.t. a definable non-standard implication) and axiomatized by a modification of the axiomatic system for L. For more details on L^* see [2].

The first-order variant $\mathrm{L}\forall^*$ of L^*, introduced in [1], is defined as usual in fuzzy [4, 5] or Rasiowa-implicative [6] logics, with predicates evaluated in L^*-algebras. In particular, a *model* for a predicate language $\mathcal{L}$ over an intended L^*-algebra $\boldsymbol{L}_*$ is $\mathbf{M} = \big(D^{\mathbf{M}}, (P^{\mathbf{M}})_{P \in \mathcal{L}}\big)$, where $D^{\mathbf{M}}$ is a crisp non-empty domain and $P^{\mathbf{M}} \colon (D^{\mathbf{M}})^n \rightarrow \boldsymbol{L}_*$ for each n-ary $P \in \mathcal{L}$. The Tarski conditions for terms and atomic formulae are defined as in $\mathrm{L}\forall$, and for propositional connectives by the truth tables above. Like the connectives of L^*, the quantifiers of $\mathrm{L}\forall^*$ also come in several families:

- The *Bochvar-style* quantifiers $\exists_{\mathrm{B}}, \forall_{\mathrm{B}}$ yield $*$ whenever an instance is $*$-valued:

$$\|(\exists_{\mathrm{B}} x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for some } a \in D^{\mathbf{M}} \\ \sup\big\{\|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} \bigm| a \in D^{\mathbf{M}}\big\} & \text{otherwise} \end{cases}$$

$$\|(\forall_{\mathrm{B}} x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for some } a \in D^{\mathbf{M}} \\ \inf\big\{\|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} \bigm| a \in D^{\mathbf{M}}\big\} & \text{otherwise.} \end{cases}$$

- The *Sobociński-style* quantifiers $\exists_{\mathrm{S}}, \forall_{\mathrm{S}}$ ignore the $*$-valued instances:

$$\|(\exists_{\mathrm{S}} x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for all } a \in D^{\mathbf{M}} \\ \sup\big\{\|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} \bigm| \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} \neq *\big\} & \text{otherwise} \end{cases}$$

$$\|(\forall_{\mathrm{S}} x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for all } a \in D^{\mathbf{M}} \\ \inf\big\{\|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} \bigm| \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} \neq *\big\} & \text{otherwise.} \end{cases}$$

- The *Kleene-style* quantifiers $\exists_{\mathrm{K}}, \forall_{\mathrm{K}}$ analogous to $\vee_{\mathrm{K}}, \wedge_{\mathrm{K}}$ can be defined as follows:

$$(\exists_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\exists_{\mathrm{B}} x)\varphi \vee_{\mathrm{K}} (\exists_{\mathrm{S}} x)\varphi, \qquad (\forall_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\forall_{\mathrm{B}} x)\varphi \wedge_{\mathrm{K}} (\forall_{\mathrm{S}} x)\varphi.$$

As usual (cf. [4, 5]), we say that a model is *safe* if all the requisite suprema and infima exist. *Validity* in a safe model is defined as truth to degree 1 under all evaluations of object variables in the model; *tautologicity* as validity in all safe models for the given language; and *entailment* as validity in all safe models validating all premises. Since L^* is Rasiowa-implicative and $\exists_{\mathrm{S}}, \forall_{\mathrm{B}}$ turn out to be interpreted by the supremum and infimum w.r.t. its matrix order, the logic $\mathrm{L}\forall^*$ can be axiomatized by adding Rasiowa's quantifier axioms [6] for $\exists_{\mathrm{S}}, \forall_{\mathrm{B}}$ to L^*. For more details on $\mathrm{L}\forall^*$ see [1].

3 Partial fuzzy modalities

As seen in the previous section, partial fuzzy logic contains several families of connectives and quantifiers that represent various modes of propagation for $*$. Because Kripke modalities can be understood as monadic quantifiers over possible worlds restricted by the accessibility relation, it is not surprising that similar situation occurs in partial fuzzy modal logic. In particular, since the semantic definitions of $\Box, \Diamond$ involve a quantifier (universal for $\Box$ and existential for $\Diamond$) and a restricting connective (implication for $\Box$ and conjunction for $\Diamond$), partial fuzzy modalities are generally indexed by two indices that specify the quantifier and the connective. For instance, $\Box_{\mathrm{BK}}$ denotes the modality of necessity defined by means of the Bochvar-style universal quantifier $\forall_{\mathrm{B}}$ and Kleene-style implication $\rightarrow_{\mathrm{K}}$.

Let $\boldsymbol{L}_*$ be an intended L^*-algebra as in Section 2. A *Kripke frame* is a structure $\mathbf{K} = (W, R)$, where W is a crisp non-empty set of possible worlds and R is a total and crisp accessibility relation on W. A *partial fuzzy Kripke model* over $\mathbf{K}$ is a pair $\mathbf{M} = (\mathbf{K}, e)$, where e is a mapping assigning to each $w \in W$ and each propositional variable p a truth value $e(w, p) \in \boldsymbol{L}_*$. Thus, the truth value of p can be undefined (i.e., equal to $*$) in some or all $w \in W$.

Let a Kripke model $\mathbf{M}$ be fixed. The truth value $\|\varphi\|_w$ of a formula φ in $\mathbf{M}$ at the world w can be defined in the obvious way for all formulae of L^*. We aim at extending the propositional language of L^* by meaningful Kripke-style modalities in $\mathbf{M}$. Due to space limitations, we shall only discuss the modalities of necessity here.

As a first step, we can consider the nine modalities $\Box_{\mathrm{XY}}$, with $\mathrm{X}, \mathrm{Y} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}\}$ referring respectively to the Bochvar, Sobociński and Kleene quantifiers and implications, defined by the following Tarski conditions:

$$\|\Box_{\mathrm{XY}}\, \varphi\|_w = (\forall_{\mathrm{X}}\, w')(Rww' \rightarrow_{\mathrm{Y}} \|\varphi\|_{w'}), \tag{1}$$

where the quantifier $\forall_{\mathrm{X}}$ ranges over W. However, three of these nine modalities collapse into one; some of the others do not respect the accessibility relation; and at least one intuitively appealing $\Box$-like modality cannot be found among them, but requires a special definition. Below we list the interesting (well-behaved) cases of $\Box_{\mathrm{XY}}$:

- *The modality* $\Box_{\mathrm{BK}}$ can be regarded as the 'Bochvar-style necessity', with $*$ acting as the absorbing element in accessible worlds, since $\|\Box_{\mathrm{BK}}\, \varphi\|_w = *$ iff $\|\varphi\|_{w'} = *$ for any $w' \in W$ such that $Rww' = 1$. (Observe that $\Box_{\mathrm{BB}}$ is not a suitable Bochvar-style necessity, as it is affected by inaccessible worlds: $\|\Box_{\mathrm{BB}}\, \varphi\|_w = *$ iff $\|\varphi\|_{w'} = *$ for *any* $w' \in W$.)

- *The modality* $\Box_{\mathrm{KK}}$ can be considered a 'Kleene-style necessity', yielding 0 if the infimum of the defined (i.e., non-$*$) values in accessible worlds is 0, and otherwise behaving as $\Box_{\mathrm{BK}}$.

- *The modality* $\Box_{\mathrm{SU}}$. Similarly, we would like to define the 'Sobociński-style necessity', i.e., the modality $\Box_{\mathrm{XY}}$ such that $\|\Box_{\mathrm{XY}}\, \varphi\|_w = *$ iff $\|\varphi\|_{w'} = *$ for all w' accessible from w, and otherwise ignoring $*$ (i.e., $\|\Box_{\mathrm{XY}}\,\varphi\|_w = \|\Box_{\mathrm{BK}}\,{\uparrow}\varphi\|_w$ otherwise). However, none of the modalities $\Box_{\mathrm{XY}}$ for $\mathrm{X}, \mathrm{Y} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}\}$ exhibits this behavior. To obtain a Sobociński-style necessity, we need to use an implication $\rightarrow_{\mathrm{U}}$ such that $0 \rightarrow_{\mathrm{U}} \beta = *$ and $1 \rightarrow_{\mathrm{U}} \beta = \beta$, for any $\beta \in \boldsymbol{L}_*$. By [2, Th. 2.4], many such implications are definable in L^*; with any of them, the modality $\Box_{\mathrm{SU}}$ possesses the desired properties described above.

- *The modality* $\Box_{\mathrm{XS}}$. With $\rightarrow_{\mathrm{S}}$ it is immaterial which of the three quantifiers is employed, since R is total and so $(Rww' \rightarrow_{\mathrm{S}} \|\varphi\|_{w'}) \neq *$; consequently, the modalities $\Box_{\mathrm{XS}}$ coincide for each $\mathrm{X} \in \{\mathrm{B}, \mathrm{K}, \mathrm{S}\}$. Since $1 \rightarrow_{\mathrm{S}} * = 0$ in $\boldsymbol{L}_*$, the modality $\Box_{\mathrm{XS}}$ behaves as the 'Bochvar-external necessity', treating $*$ as falsity on the accessible worlds: $\|\Box_{\mathrm{XS}}\,\varphi\|_w = 0$ if $\|\varphi\|_{w'} = *$ for some w' such that $Rww' = 1$. It can be easily shown that $\|\Box_{\mathrm{XS}}\,\varphi\|_w = \|\Box_{\mathrm{BK}}\,{\downarrow}\varphi\|_w \neq *$. (Cf. the Bochvar-external universal quantifier of [1].)

Using the Tarski conditions (1) in partial fuzzy Kripke models, it is straightforward to define the local and global consequence relations for the extension of partial fuzzy logic L^* by the aforementioned modalities. (Note that the semantic value assignment $\|\cdot\|$ can equivalently be understood as a syntactic translation of the modal language into $\mathrm{L}\forall^*$; cf. [4, §8.3].) In the talk we will discuss the partial fuzzy modalities in more detail; show their basic properties; present examples of valid and invalid modal formulae and rules of partial fuzzy modal logic; and discuss possible generalizations and applications of the apparatus.

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MacNeille transferability of finite lattices*

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A lattice L is *transferable* if whenever there is a lattice embedding $h\colon L \to \mathrm{Idl}(K)$ of L into the ideal lattice $\mathrm{Idl}(K)$ of a lattice K, then there is a lattice embedding $h'\colon L \to K$. It is *sharply transferable* if h' can be chosen so that $h'(x) \in h(y)$ iff $x \leq y$. If we restrict K to belong to some class of lattices $\mathcal{K}$, we say L is *transferable* for $\mathcal{K}$. The notion has a long history, beginning with [8, 5], and remains a current area of research [11]. For a thorough account of the subject, see [9], but as a quick account, among the primary results in the area are the following.

Theorem 1. (see [9, pp. 502–503],[11]) *A finite lattice is transferable for the class of all lattices iff it is projective, and in this case it is sharply transferable.*

Theorem 2. (see [5, 10]) *Every finite distributive lattice is sharply transferable for the class of all distributive lattices.*

Our purpose here is to introduce, and make the first steps towards, an analogous study of MacNeille transferability.

For a lattice K, we use $\overline{K}$ for the MacNeille completion of K. We also consider lattices with one or both bounds as part of their basic type. For $\tau \subseteq \{\wedge, \vee, 0, 1\}$, a *$\tau$-lattice* is a lattice, or lattice with one or both bounds, whose basic operations are of type τ. A *τ-homomorphism* is a homomorphism with respect to this type, and a *τ-embedding* is a one-one τ-homomorphism.

Definition 3. Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$, L be a τ-lattice, and $\mathcal{K}$ a class of τ-lattices. Then L is *τ-MacNeille transferable* for $\mathcal{K}$ if for any τ-embedding $h\colon L \to \overline{K}$ where $K \in \mathcal{K}$, there is a τ-embedding $h'\colon L \to K$. We say L is *sharply τ-transferable* for $\mathcal{K}$ if h' can be chosen so that $h'(x) \leq h(y)$ iff $x \leq y$ for all $x, y \in L$. We use the terms *MacNeille transferable* and *sharply MacNeille transferable* when $\tau = \{\wedge, \vee\}$.

There are obvious examples of lattices L that are MacNeille transferable for the class of all lattices. Any finite chain, and the 4-element Boolean lattice provide such examples. Infinite chains can be problematic, as is seen by a simple cardinality argument for L the chain of real numbers and K the chain of rational numbers. Such difficulties arise also with the traditional study of transferability using ideal lattices. Therefore we here restrict our attention to the case where L is finite.

Characterizations as in Theorems 1 and 2 are beyond our scope. But we do provide a number of results, both positive and negative, that begin to frame the problem. Among easy results are the following:

Theorem 4. *Any lattice which is MacNeille transferable for a class of lattices containing all distributive lattices is distributive.*

Theorem 5. *Every finite projective distributive lattice is MacNeille transferable for the class of lattices whose MacNeille completions are distributive.*

*The results presented here can be found in the forthcoming paper [3].

We obtain stronger positive results for bi-Heyting algebras, namely by showing that the MacNeille completion of a bi-Heyting algebra of finite width is a bounded sublattice of its ideal lattice we obtain:

Theorem 6. *Every finite distributive lattice is MacNeille transferable for the class of bi-Heyting algebras.*

Among further results, we show that that there are finite non-projective distributive lattices MacNeille transferable—but not sharply MacNeille transferable—for the class of distributive lattices, and therefore that the converse of Theorem 5 fails. In fact, we provide an infinite family of finite non-projective distributive lattices that are MacNeille transferable for the class of Heyting algebras whose dual spaces have finite width.

As negative results we give an example of a finite distributive lattice that is not MacNeille transferable for the class of lattices whose MacNeille completions are distributive, and in the case where MacNeille transferability is extended to include the bounds, we show:

Theorem 7. *If L is a non-trivial finite distributive lattice with a non-trivial complemented element then L is not $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for the class of Heyting algebras.*

Finally, we note that for any finite lattice L and any $\tau \subseteq \{\wedge, \vee, 0, 1\}$ there is a universal clause, i.e., Π_1-sentence, $\rho_\tau(L)$, in the language τ, expressing the property of not having a τ-subalgebra isomorphic to L. Consequently, determining if a finite lattice L is τ-MacNeille transferable is equivalent to determining if the universal clause $\rho_\tau(L)$ is preserved under MacNeille completions. From this perspective some of our results are interderivable with special cases of results obtained by Ciabattoni et al. [4].

Universal classes of Heyting algebras axiomatised by universal clauses of the form $\rho_\tau(L)$, for $\tau = \{\wedge, \vee, 0, 1\}$, correspond to so-called stable intermediate logics [1, 2]. Our results provide an infinite family of stable universal classes of Heyting algebras that are closed under MacNeille completions. Due to the result of [7], this implies that these stable universal classes are also closed under canonical extensions and therefore by [6] that the corresponding intermediate logics are canonical.

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Epistemic MV-Algebras

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Abstract

We generalize the notion of monadic MV-algebras to that of Epistemic MV-algebras. As monadic MV-algebras serve as algebraic models of modal logic $S5(Ł)$, we propose epistemic MV-algebras as algebraic models of modal system $KD45(Ł)$. The main contributions of this presentation are two: 1) we offer a characterization of epistemic MV-algebras as pairs of MV-algebras $(\mathbf{A}, \mathbf{B})$ where $\mathbf{B}$ is a special case of a relatively complete subalgebra of $\mathbf{A}$ called c-relatively complete. We also give a necessary and sufficient condition for a subalgebra to be c-relatively complete; 2) we study the complex MV-algebras over MV-chains and we determinate its connection with a simplified version of Kripke semantics. Furthermore, we analyse the relation between complex MV-chains and the modal logic given by an MV-chain studied by Bou et al. in [2].

1 Introduction

MV-algebras are the equivalent algebraic semantics of infinite-valued Łukasiewicz logic (see [4]). It is widely known that they coincide with involutive BL-algebras (see [6]). In other words, the variety of MV-algebras is term-equivalent to the subvariety of BL that satisfies $\neg\neg a = a$. In the last decade, under different forms and contexts, modal extensions of Łukasiewicz logic have appeared in the literature for different reasoning modelling purposes. For instance, in [7], the authors present an algebraic approach to some Łukasiewicz modal logics and its relationship with a crisp-relational semantics, i.e. where the accessibility relation only takes two values. Another example appears in [2], where it was shown that the n-valued Łukasiewicz modal logic is axiomatizable. In [5], the authors partially extend these results when replacing the underlying logic $Ł_n$ by the infinite-valued Łukasiewicz modal logic (with rational truth constants in the language). In fact, they prove that the tautologies of the infinite-valued Łukasiewicz modal logic are characterized as the intersection of the tautologies of every n-valued Łukasiewicz modal logic with $n \in \mathbb{N}$. Unfortunately, it is not immediate to find in this way an axiomatization of the infinite-valued Łukasiewicz modal logic.

Finally, it is worth mentioning the infinite-valued Łukasiewicz modal logics studied by Hájek in Chapter 8 of [6]. In particular, the fuzzy variants $S5(Ł)$ and $KD45(Ł)$ of the well-known classical modal logics $S5$ and $KD45$, respectively, are studied. In fact, the first logic is axiomatized but the axiomatization of the second one is left as an open problem.

We want to attack this open problem in a novel way, by proposing a possible algebraic semantics, which is obtained by extending MV-algebras (the algebraic models of Łukasiewicz logic) by an operator that models possibility [1]. To achieve our aim, we introduce a generalization of Monadic MV-algebras defined and studied by Rutledge in his Ph.D thesis ([8]) which we call Epistemic MV-algebras (EMV-algebras). This generalization resembles what is done with monadic Boolean algebras and Pseudomonadic algebras in [1].

Throughout this presentation, we assume that the reader is acquainted with basic notions concerning MV-algebras. For details about MV-algebras see [4].

2 Epistemic MV-algebras

Definition 1. *An algebra* $\mathbf{A} = \langle A, \oplus, \neg, \exists, 0\rangle$ *of type* $(2, 1, 1, 0)$ *is called an Epistemic* MV-*algebra (an* EMV-*algebra for short) if* $\langle A, \oplus, \neg, 0\rangle$ *is a* MV-*algebra that also satisfies:*

(EMV$\exists$)	$\exists 0$	$=$	0	(EMV3)	$\exists a \odot \exists\neg\exists a$	$=$	0
(EMV1)	$\exists\neg a \oplus \exists a$	$=$	1	(EMV4)	$\exists(a \vee b)$	$=$	$\exists a \vee \exists b$
(EMV2)	$\neg\exists(a \odot \neg\exists b)$	$=$	$\exists a \to \exists b$	(EMV5)	$\exists(a \odot \exists b)$	$=$	$\exists a \odot \exists b$

where the operations $\odot$, $\to$, $\vee$, *and* $\wedge$ *are defined as follows:* $a \odot b := \neg(\neg a \oplus \neg b)$, $a \to b := \neg a \oplus b$, $a \vee b := \neg(\neg x \oplus b) \oplus b$ *and* $a \wedge b := a \odot (a \to b)$.

Epistemic MV-algebras form a variety that we will denote by $\mathbb{EMV}$, and for simplicity, if $\mathbf{A}$ is a MV-algebra and we enrich it with an epistemic structure, we denote the resulting algebra by $\langle \mathbf{A}, \exists\rangle$. In an EMV-algebra, we can define $\forall : A \mapsto A$ by $\forall a = \neg\exists\neg a$.

We recall that $\mathbf{A} = \langle A, \odot, \to, 0\rangle$ is a BL-algebra which satisfies the equation $(a \to 0) \to 0 = a$. On the other hand, if we take a BL-algebra $\mathbf{B} = \langle B, \odot, \to, 0\rangle$ satisfying the equation $(a \to 0) \to 0 = a$ and for each $a, b \in B$ we define $\neg a = a \to 0$ and $a \oplus b = \neg(\neg a \odot \neg b)$, then $\mathbf{B} = \langle B, \oplus, \neg, 0\rangle$ is an MV-algebra.

In [3], the following definition of Epistemic BL-algebras is proposed:

Definition 2. *An algebra* $\mathbf{A} = \langle A, \vee, \wedge, \odot, \to, \forall, \exists, 0, 1\rangle$ *of type* $(2, 2, 2, 2, 1, 1, 0, 0)$ *is called an Epistemic BL-algebra (an* EBL-*algebra for short) if* $\langle A, \vee, \wedge, \odot, \to, 0, 1\rangle$ *is a* BL-*algebra that also satisfies:*

(E$\forall$)	$\forall 1$	$=$	$1,$	(E4)	$\exists a \to \forall\exists a$	$=$	$1,$
(E$\exists$)	$\exists 0$	$=$	$0,$	(E4A)	$\forall(a \wedge b)$	$=$	$\forall a \wedge \forall b,$
(E1)	$\forall a \to \exists a$	$=$	$1,$	(E4B)	$\exists(a \vee b)$	$=$	$\exists a \vee \exists b,$
(E2)	$\forall(a \to \forall b)$	$=$	$\exists a \to \forall b,$	(E5)	$\exists(a \odot \exists b)$	$=$	$\exists a \odot \exists b.$
(E3)	$\forall(\forall a \to b)$	$=$	$\forall a \to \forall b,$				

Epistemic BL-algebras form a variety that we will denote by $\mathbb{EBL}$. In our presentation, we are going to show that the subvariety of $\mathbb{EBL}$ determined by the equation $\neg\neg a = a$ is term-equivalent to the variety of $\mathbb{EMV}$. Due to this fact, we can use some of the results of [3] to study $\mathbb{EMV}$. For instance, we prove the following:

Theorem 1. *If* $\mathbf{A} \in \mathbb{EMV}$, *then* $\exists A$ *is closed under the operations of* $\mathbf{A}$, *thus* $\exists\mathbf{A}$ *is a subalgebra of* $\mathbf{A}$.

[1] The necessity operator could be definable in this case because of the involution.

Given an EMV-algebra $\langle \mathbf{A}, \exists \rangle$, if the set

$$\{a \in A : \forall a = 1\} \subsetneq A$$

has a least element c, then c will be called *focal element* of $\mathbf{A}$. This focal element satisfies:

$$c = \min_{a \in A}\{(a \to \exists a)\}$$

According to that, we are going to say that an EMV-algebra $\mathbf{A}$, is a c-EMV-algebra, if the focal element c exists in A. It is worth pointing out that this focal element plays an important role, since it allows us to recover the unary operator $\exists$, as the following theorem shows

Theorem 2. *Let $\mathbf{A}$ be a* c-EMV*-algebra and let $\mathbf{B}$ be the subalgebra given by Theorem 1, then*

$$\exists a = \min\{b \in B : c \odot a \leq b\}$$

Moreover, we are going to show under which conditions an c-EMV-algebra can be defined from a MV-algebra $\mathbf{A}$ and one of its subalgebras $\mathbf{B}$.

Definition 3. *Let $\mathbf{A}$ be a* MV*-algebra, $\mathbf{B}$ a subalgebra of $\mathbf{A}$ and $c \in A$. We say that the pair $(\mathbf{B}, c)$ is a c-*relatively complete subalgebra*, if the following conditions hold:*

(e1) *For every $a \in A$, the subset $\{b \in B : c \odot a \leq b\}$ has a least element.*

(e2) $\{a \in A : c^2 \leq a\} \cap B = \{1\}$.

Theorem 3. *Given a* MV*-algebra $\mathbf{A}$ and a c-relatively complete subalgebra $(\mathbf{B}, c)$, if we define on $\mathbf{A}$ the operations:*

$$\exists a := \min\{b \in B : c \odot a \leq b\}, \tag{1}$$

then $\langle \mathbf{A}, \exists \rangle$ is a c-EMV*-algebra such that $\exists \mathbf{A} = \mathbf{B}$. Conversely, if $\mathbf{A}$ is a* c-EMV*-algebra, then $(\exists A, c)$ is a c-relatively complete subalgebra of $\mathbf{A}$.*

Last theorem shows that there exists a one-to-one correspondence between c-EMV-algebras and the pairs $(\mathbf{A}, \mathbf{B})$ where $(\mathbf{B}, c)$ is a c-relatively complete subalgebra of $\mathbf{A}$.

To finish our presentation, we will try to explain the relation between this algebraic semantics and the well known Kripke semantics. Our goal will then be to establish a connection between EMV-algebras and the modal logic given by a finite MV-chain studied by [2].

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General Neighborhood and Kripke Semantics for Modal Many-Valued Logics*

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In modal extensions of classical logic, Kripke frames and Scott–Montague neighborhood frames (see e.g. [21, 26]) provide two, widely used, different kinds of frame semantics; the former is intended for normal modal logics (i.e. extensions of K), while the latter works as well for non-normal modal logics extending the weaker system E. However, not all logics extending these basic logics are complete with respect to a class of corresponding frames.

This problem has been amended by enhancing frames with collections of *admissible* sets of worlds (i.e., prescribed sets that *are allowed to be* assigned to a formula by a valuation). Such generalized semantics has proved to be more versatile as it guarantees completeness theorems for all modal logics over K or E respectively [9].

Many-valued propositional logics expanded with modal operators have been a topic of interest since Fitting's initial works [13, 14], later continued by Petr Hájek and others in the field of Mathematical Fuzzy Logic (see e.g. [3–8, 10, 16–20, 27, 28]). Such modal logics are typically endowed with a *many-valued* Kripke semantics, i.e., a semantics over a given algebra (usually a residuated lattice) whose elements are used as possible values of formulas in each world and/or degrees of accessibility from one world to another.[1] As in the classical case, Kripke-style semantics has its limitations; but the situation here is even more complicated: indeed, to find the axiomatization of the class of all frames often becomes difficult.

An alternative frame semantics, with wider applicability and better logical properties, can be proposed. Following earlier works of Godo and Rodríguez [23, 24], in this talk we present *the first steps* towards a general theory of Scott–Montague semantics for many-valued logics.

We introduce (general) Scott–Montague frames over an arbitrary $\mathrm{FL}_{\mathrm{ew}}$-algebra $\boldsymbol{A}$ with operators. We discuss its relations with the (general) Kripke semantics and prove several completeness results for the global consequence relation over these frames, in particular for all extensions of the logic E over the logic of $\boldsymbol{A}$, provided that the logic of $\boldsymbol{A}$ is finitary (these results can be seen in full details in the recent papers [11, 12]).

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[1] It should be mentioned that there is another tradition in the study of modal extensions of non-classical logics which already have a certain 'classical' frame semantics (classical in the sense that both accessibility relations and valuation functions are two-valued), typical examples being the intuitionistic, relevant and other substructural logics. In this approach, modalities are modeled using additional *classical* accessibility relations/neigbourhood functions (see e.g. [2, 15, 25] or the corresponding chapter of [22]). Even though there are certain relations between both approaches stemming from algebra/frame dualities, the results of this stream of research are not directly relevant to the framework presented here.

1. General $\boldsymbol{A}$-neighborhood frames and the completeness theorem

A *general $\boldsymbol{A}$-neighborhood frame*, a gSM($\boldsymbol{A}$)-frame for short, is a tuple $\mathfrak{F} = \langle W, N_\Box, N_\Diamond, S\rangle$ such that $W \neq \emptyset$, S is a subuniverse of $\boldsymbol{A}^W$, and $N_\Box, N_\Diamond \colon S \to S$ are the *neighborhood functions.*

Furthermore, a *general $\boldsymbol{A}$-neighborhood model* is a tuple $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$, where $\mathfrak{F} = \langle W, N^\Box, N^\Diamond, S\rangle$ is an gSM($\boldsymbol{A}$)-frame and $V\colon Var \to S$ is an *admissible* valuation. The valuation is extended to all formulas $V^{\mathfrak{M}}\colon Fm_{\mathcal{L}}^{\Box\Diamond} \to S$, defining:

$$
\begin{aligned}
V^{\mathfrak{M}}(\Box\varphi) &= N_\Box(V^{\mathfrak{M}}(\varphi)),\\
V^{\mathfrak{M}}(\Diamond\varphi) &= N_\Diamond(V^{\mathfrak{M}}(\varphi)),\\
V^{\mathfrak{M}}(\circ(\varphi_1,\dots,\varphi_n)) &= \circ^{\boldsymbol{A}^W}(V^{\mathfrak{M}}(\varphi_1), V^{\mathfrak{M}}(\varphi_2),\dots,V^{\mathfrak{M}}(\varphi_n)), \qquad \text{for any } n\text{-ary } \circ \in \mathcal{L}.
\end{aligned}
$$

A general $\boldsymbol{A}$-neighborhood frame is *full* if $S = A^W$ and, then, it can be identified with just the tuple $\langle W, N_\Box, N_\Diamond\rangle$ which we call simply an '$\boldsymbol{A}$-neighborhood frame'. By (g)SM($\boldsymbol{A}$) we also denote the class all of (general) $\boldsymbol{A}$-neighborhood frames.

We say that φ is a *(global) consequence* of Γ w.r.t. a class $\mathbb{F}$ of general $\boldsymbol{A}$-neighborhood frames, $\Gamma \models^g_{\mathbb{F}} \varphi$ in symbols, if for each $\mathfrak{F} \in \mathbb{F}$ and each model $\mathfrak{M}$ over $\mathfrak{F}$ we have:

$$\text{if } V^{\mathfrak{M}}(\gamma) = W \text{ for each } \gamma \in \Gamma, \text{ then also } V^{\mathfrak{M}}(\varphi) = W.$$

Since we do not deal with local consequence here, we omit the prefix 'global' and its superindex.

Recall that the logic of an algebra $\boldsymbol{A}$, denoted as $\models_{\boldsymbol{A}}$, need not be finitary. Therefore, we define the logic $\mathrm{E}_{\boldsymbol{A}}$ as the expansion of the *finitary companion*[2] of $\models_{\boldsymbol{A}}$ by two additional rules:

$$\varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi \qquad \text{and} \qquad \varphi \leftrightarrow \psi \vdash \Diamond\varphi \leftrightarrow \Diamond\psi.$$

By gSM(L) we denote the class of all gSM($\boldsymbol{A}$)-frames $\mathfrak{F}$ sound w.r.t. L, i.e., such that for each $\Gamma \cup \{\varphi\}$ we have $\Gamma \models_{\mathfrak{F}} \varphi$ whenever $\Gamma \vdash_{\mathrm{L}} \varphi$. Note that we can easily prove that gSM($\mathrm{E}_{\boldsymbol{A}}$) = gSM($\boldsymbol{A}$).

Theorem 1. *Let $\boldsymbol{A}$ be an $\mathrm{FL_{ew}}$-algebra with operators such that $\models_{\boldsymbol{A}}$ is finitary and let* L *be an extension of $\mathrm{E}_{\boldsymbol{A}}$. Then, for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}^{\Box\Diamond}$, we have:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathrm{gSM(L)}} \varphi.$$

In the particular case of the basic logic $\mathrm{E}_{\boldsymbol{A}}$ we actually have completeness with respect to all full SM($\boldsymbol{A}$)-frames, i.e., if $\models_{\boldsymbol{A}}$ is finitary we have $\Gamma \vdash_{\mathrm{E}_{\boldsymbol{A}}} \varphi$ iff $\Gamma \models_{\mathrm{SM}(\boldsymbol{A})} \varphi$. Also, in this case, if we restrict to finite sets Γ the equivalence holds even without assuming that $\models_{\boldsymbol{A}}$ is finitary.

2. Relation with Kripke semantics

A *general $\boldsymbol{A}$-Kripke frame*, a gKR($\boldsymbol{A}$)-frame for short, is a tuple $\mathfrak{F} = \langle W, R, S\rangle$ such that $W \neq \emptyset$, R is a binary $\boldsymbol{A}$-valued relation, and S is a subuniverse of $\boldsymbol{A}^W$ closed under functions $R^\Box, R^\Diamond\colon A^W \to A^W$ defined for each $X \in A^W$ as[3]

$$
\begin{aligned}
R^\Box(X) = &\ \{w \mid R[w] \subseteq X\} &= \{w \mid \bigwedge_{v\in W} (wRv \to^{\boldsymbol{A}} v \in X)\},\\
R^\Diamond(X) = &\ \{w \mid R[w] \between X\} &= \{w \mid \bigvee_{v\in W} (wRv \,\&^{\boldsymbol{A}}\, v \in X)\}.
\end{aligned}
$$

[2]We define $\Gamma \vdash_{\mathrm{FC}(A)} \varphi$ iff there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models_{\boldsymbol{A}} \varphi$.

[3]We view the elements of A^W as $\boldsymbol{A}$-valued fuzzy sets (e.g., we write $w \in X$ instead of $X(w)$) and the operations on the complex product $\boldsymbol{A}^W$ as operations on fuzzy sets. Utilizing the formalism of Fuzzy Class Theory [1], we use the comprehension terms to describe elements of A^W, and the well-known graded subsethood and overlap relations.

Furthermore, a *general $\boldsymbol{A}$-Kripke model* is a tuple $\mathcal{M} = \langle \mathfrak{F}, V \rangle$, where $\mathfrak{F} = \langle W, R, S \rangle$ is a gKR($\boldsymbol{A}$)-frame and $V \colon Var \to S$ is an *admissible* valuation. The valuation is extended to all formulas $V^{\mathcal{M}} \colon Fm_{\mathcal{L}}^{\Box\Diamond} \to S$, defining:

$$\begin{aligned} V^{\mathcal{M}}(\Box\varphi) &= R^{\Box}(V^{\mathcal{M}}(\varphi)), \\ V^{\mathcal{M}}(\Diamond\varphi) &= R^{\Diamond}(V^{\mathcal{M}}(\varphi)), \\ V^{\mathcal{M}}(\circ(\varphi_1, \dots, \varphi_n)) &= \circ^{\boldsymbol{A}^W}(V^{\mathcal{M}}(\varphi_1), V^{\mathcal{M}}(\varphi_2), \dots, V^{\mathcal{M}}(\varphi_n)), \qquad \text{for any } n\text{-ary } \circ \in \mathcal{L}. \end{aligned}$$

We say that a gSM($\boldsymbol{A}$)-frame $\mathfrak{F} = \langle W, N^{\Box}, N^{\Diamond}, S \rangle$ is *equivalent* to a gKR($\boldsymbol{A}$)-frame $\mathfrak{F}' = \langle W, R, S \rangle$ if for each admissible valuation V and each formula φ we have $V^{\langle \mathfrak{F}, V \rangle}(\varphi) = V^{\langle \mathfrak{F}', V \rangle}(\varphi)$.

The relationship between full $\boldsymbol{A}$-Kripke and full $\boldsymbol{A}$-neighborhood frames mimics that of classical logic (as before, we identify full general $\boldsymbol{A}$-Kripke frames with simple $\boldsymbol{A}$-Kripke frames).

An SM($\boldsymbol{A}$)-frame $\mathfrak{F} = \langle W, N^{\Box}, N^{\Diamond} \rangle$ is *augmented* if, for each $w \in W$, there is a unique $C_w \in A^W$ such that for each $X \in A^W$,

$$C_w \subseteq X = w \in N^{\Box}(X) \qquad \text{and} \qquad C_w \between X = w \in N^{\Diamond}(X).$$

Theorem 2. *Let $\boldsymbol{A}$ be a complete* $\mathrm{FL_{ew}}$*-algebra with operators.*

(a) If $\mathfrak{F} = \langle W, R \rangle$ is a K($\boldsymbol{A}$)*-frame, then $\mathfrak{F}_n = \langle W, R^{\Box}, R^{\Diamond} \rangle$ is an augmented* SM($\boldsymbol{A}$)*-frame equivalent to $\mathfrak{F}$.*

(b) If $\mathfrak{F} = \langle W, N^{\Box}, N^{\Diamond} \rangle$ is an augmented SM($\boldsymbol{A}$)*-frame, then $\mathfrak{F}_k = \langle W, R_{\mathfrak{F}} \rangle$, where $R_{\mathfrak{F}}wv = v \in C_w$ is a* K($\boldsymbol{A}$)*-frame equivalent to $\mathfrak{F}$.*

(c) For each K($\boldsymbol{A}$)*-frame $\mathfrak{F}$, we have: $\mathfrak{F} = (\mathfrak{F}_n)_k$.*

(d) For each augmented SM($\boldsymbol{A}$)*-frame $\mathfrak{F}$, we have: $\mathfrak{F} = (\mathfrak{F}_k)_n$.*

Note one fundamental difference between general Kripke and neighborhood frames: for each gKR($\boldsymbol{A}$)-frame $\mathfrak{F} = \langle W, R, S \rangle$, the reduct $\langle W, R \rangle$ is an $\boldsymbol{A}$-Kripke frame. In neighborhood frames the same trick does not work as the functions used to interpret the modalities are defined only for admissible $\boldsymbol{A}$-valued sets and, thus, for each gSM($\boldsymbol{A}$)-frame $\mathfrak{F} = \langle W, N^{\Box}, N^{\Diamond}, S \rangle$ there could be many different SM($\boldsymbol{A}$)-frames $\mathfrak{F}^f = \langle W, N_f^{\Box}, N_f^{\Diamond} \rangle$ such that $N_f^{\Box}|S = N^{\Box}$ and $N_f^{\Diamond}|S = N^{\Diamond}$.

We can overcome this problem by using suitable notions of *augmented* gSM($\boldsymbol{A}$)-frame and *tight* gKR($\boldsymbol{A}$)-frame. We prove that to each gKR($\boldsymbol{A}$)-frame $\mathfrak{F}$ we can assign an equivalent augmented gSM($\boldsymbol{A}$)-frame $\mathfrak{F}_{gn}$. Dually, to each augmented gSM($\boldsymbol{A}$)-frame we assign an equivalent tight gKR($\boldsymbol{A}$)-frame $\mathfrak{F}_{gk}$. Furthermore, for each augmented gSM($\boldsymbol{A}$)-frame $\mathfrak{F}$, we have: $\mathfrak{F} = (\mathfrak{F}_{gk})_{gn}$; and for each *tight* gKR($\boldsymbol{A}$)-frame $\mathfrak{F}$, we have: $\mathfrak{F} = (\mathfrak{F}_{gn})_{gk}$ (without the tightness assumption we would only obtain that $\mathfrak{F}$ is equivalent to $(\mathfrak{F}_{gn})_{gk}$).

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Hereditarily Structurally Complete Positive Logics

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Abstract

We give a description of all hereditarily structurally complete positive logics.

1 Introduction

The notion of admissible rule evolved from the notion of auxiliary rule: if a formula B can be derived from a set of formulas $A_1, \dots, A_n$ in a given calculus (deductive system) DS, one can shorten derivations by using a rule $A_1, \dots, A_n/B$. The application of such a rule does not extend the set of theorems, i.e. such a rule is admissible (permissible). In [9, p.19] P. Lorenzen called the rules not extending the class of the theorems "zulässig," and the latter term was translated as "admissible," the term we are using nowadays. In [10] Lorenzen also linked the admissibility of a rule to existence of an elimination procedure.

Independently, P. S. Novikov, in his lectures on mathematical logic, had introduced the notion of a derived rule: a rule $\mathcal{A}_1, \dots, \mathcal{A}_n/\mathcal{B}$, where $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$ are variable formulas of some type, is derived in a calculus DS if $\vdash_{\mathsf{DS}} \mathcal{B}$ holds every time when $\vdash_{\mathsf{DS}} \mathcal{A}_1, \dots, \vdash_{\mathsf{DS}} \mathcal{A}_n$ hold (see [13, p. 30][1]). And he distinguished between two types of the derived rules: a derived rule is strong, if $\vdash_{\mathsf{DS}} \mathcal{A}_1 \to (\mathcal{A}_2 \to \dots (\mathcal{A}_n \to \mathcal{B}) \dots)$ holds, otherwise a derived rule is weak.

For classical propositional calculus (CPC), the use of admissible rules is merely a matter of convenience, because every admissible in CPC rule $A_1, \dots, A_n/B$ is derivable, that is $A_1, \dots, A_n \vdash_{\mathsf{CPC}} B$ (see, for instance [1]). It was observed by R. Harrop in [8] that the rule $\neg p \to (q \vee r)/(\neg p \to q) \vee (\neg p \to r)$ is admissible in intuitionistic propositional logic (Int), but is not derivable: the corresponding formula is not a theorem of Int. Later, in mid 1960s, A. V. Kuznetsov observed that the rule $(\neg\neg p \to p) \to (p \vee \neg p)/((\neg\neg p \to p) \to \neg p) \vee ((\neg\neg p \to p) \to \neg\neg p)$ is also admissible in Int, but not derivable. Another example of an admissible for IPC not derivable rule was found in 1971 by G. Mints (see [11]): the following rule is admissible but not derivable in Int

$$(p \to q) \to (p \vee r)/((p \to q) \to p) \vee ((p \to q) \to r). \tag{1}$$

Naturally, the question about admissibility of rules in the intermediate logics (the extensions of Int understood as a set of theorems) and their fragments arose. And in [16] T. Prucnal proved that $\to$-fragment of any intermediate logic, is ***structurally complete***, i.e. these fragments do not have admissible not-derivable rules. If a logic and all its extensions are structurally complete, such a logic is ***hereditarily structurally complete*** (*HSCpl*).

In terms of hereditary structural completeness, the aforementioned Prucnal's result can be rephrased as follows: the $\to$-fragment of Int is hereditarily structurally complete. Curiously enough, the implication-negation (or implication-falsity) fragment of Int is not structurally complete (see [18]). In [3], Cintula and Metcalfe described hereditarily structurally complete

[1] This book was published in 1977, but it is based on the notes of a course that P.S. Novikov taught in 1950s; A.V. Kuznetsov was recalling that P.S. Novikov had used the notion of derivable rule much earlier, in this lectures in 1940s.

implication-negation fragments of the intermediate logics, and they proved that there is the smallest (hereditarily) structurally complete implication-negation fragment. The situation with hereditary structural completeness of fragments intermediate logics is summarized in Table 1 where "Not HSCpl" means that not all intermediate logics are HSCpl with respect to a particular fragment.

Fragment		Reference
$\{\to\}$	*HSCpl*	[16]
$\{\to,\bot\}$	Not *HSCpl* (the smallest *HSCpl* fragment exists)	[3]
$\{\to,\land\}$	*HSCpl*	[12]
$\{\to,\land,\bot\}$	*HSCpl*	[18]
$\{\to,\land,\lor\}$	Not *HSCpl* (the smallest *HSCpl* fragment exists)	this abstract
$\{\to,\land,\lor,\bot\}$	Not *HSCpl* (the smallest *HSCpl* fragment exists)	[4]

Table 1: Structural Completeness of Fragments of Intermediate Logics

An interesting sufficient condition for positive predicate logic to be hereditarily structurally complete was proved by Dzik (cf. [5, Theorem 3]).

Let us make an easy but nonetheless curious observation: the positive fragment of the Medvedev Logic ML is not structurally complete, in particular, rule (1) is admissible but not derivable in ML. This raises the following question.

Problem 1. Is every structurally complete positive logic hereditarily structurally complete?

2 Main Results

We consider intuitionistic propositional logic Int with connectives $\land,\lor,\to,\bot,\top$. The (propositional) formulas which have no occurrences of $\bot$ are called ***positive***. Clearly, the set Int^+ of all positive formulas from Int is closed under the application of rules modus ponens (denoted by MP) and (simultaneous) substitution (denoted by Sb). The set of all extensions of Int^+, closed under MP and Sb is denoted by ExtInt^+, and we refer to members of ExtInt^+ as ***positive logics***.

Algebraic models for positive logics are Brouwerian algebras[2]: an algebra $\langle\mathsf{A};\land,\lor,\to,\mathbf{1}\rangle$, where $\langle\mathsf{A};\land,\lor,\mathbf{1}\rangle$ is a distributive lattice with a greatest element $\mathbf{1}$, and $\to$ is a relative pseudo-complementation, is a ***Brouwerian algebra***.

In a regular way we define validness of a formula in a given algebra: a formula A is ***valid*** in a given (Brouwerian) algebra $\mathbf{A}$ (in symbols, $\mathbf{A}\models A$) if $v(A)=\mathbf{1}$ for every valuation v in $\mathbf{A}$. With each logic $\mathsf{L}\in\mathsf{ExtInt}^+$ we associate a set $\mathsf{V}(\mathsf{L})$ of all algebras in which every formula from L is valid:

$$\mathsf{V}(\mathsf{L}) := \{\mathbf{A}\mid\mathbf{A}\models A,\ \text{for every } A\in\mathsf{L}\}.$$

The class $\mathsf{V}(\mathsf{L})$ forms a variety.

On the other hand, every variety $\mathcal{V}$ of Brouwerian algebras defines a positive logic:

$$\mathsf{L}(\mathcal{V}) := \{A\mid\mathbf{A}\models A \text{ for every } \mathbf{A}\in\mathcal{V}\}.$$

[2]We follow [6] and call these algebras Brouwerian. Some authors are using different names, for instance, implicative lattices [14], lattice with pseudocomplementation [17].

Theorem 2.1 (Main Theorem). *A positive logic L is hereditarily structurally complete if and only if Brouwerian algebras $\mathbf{S}_1$ and $\mathbf{S}_2$ depicted in Fig. 1. are not models for L.*

It is known (see e.g. [15]) that logic L is *HSCpl* if and only if variety $\mathsf{V}(\mathsf{L})$ is ***primitive***, that is every subquasivariety of $\mathsf{V}(\mathsf{L})$ is a variety. Thus, the above theorem is equivalent to the following statement.

Theorem 2.2 (Main Theorem: Algebraic Version). *A variety $\mathcal{V}$ of Brouwerian algebras is primitive if and only if $\mathbf{S}_1, \mathbf{S}_2 \notin \mathcal{V}$ (see Fig. 1).*

A proof of the above theorem follows from the following lemmas.

Lemma 2.3. *Any variety that contains algebra $\mathbf{S}_1$ or $\mathbf{S}_2$ is not primitive.*

Recall that a variety $\mathcal{V}$ is said to be ***locally finite*** if any finitely generated algebra from $\mathcal{V}$ is finite. The following lemma gives an easy sufficient condition of local finiteness.

Lemma 2.4. *Any variety of Brouwerian algebras not containing algebra $\mathbf{S}_1$ is locally finite.*

Recall also that an algebra $\mathbf{A}$ from a class of algebras $\mathcal{K}$ is ***weakly projective*** (or ***primitive*** [2]) in $\mathcal{K}$ if $\mathbf{A} \in \mathbb{S}\mathbf{B}$ for every algebra $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A} \in \mathbb{H}\mathbf{B}$.

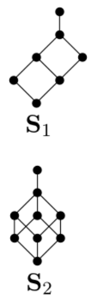

Figure 1: Non-projective Brouwerian Algebras.

Lemma 2.5. *In any variety of Brouwerian algebras not containing algebras $\mathbf{S}_1, \mathbf{S}_2$, every finite subdirectly irreducible algebra is weakly projective.*

It is clear that Lemma 2.3 is just a necessary condition of hereditary structural completeness, while the sufficient condition follows immediately from Lemmas 2.4, 2.5 and the following Proposition (see [7, Proposition 5.1.24]).

Proposition 2.6. *Let $\mathcal{V}$ be a locally finite variety of a finite signature. Then, $\mathcal{V}$ is primitive if and only if every finite subdirectly irreducible algebra from $\mathcal{V}$ is weakly projective in the set of all finite algebras from $\mathcal{V}$.*

Corollary 2.7 (Main Corollary). *The following hold:*

(a) There is the smallest HSCpl positive logic and it is finitely axiomatized;

(b) The set of all HSCpl positive logics is countable;

(c) Every HSCpl positive logic is finitely axiomatizable;

(d) There are infinitely many HSCpl intermediate logics whose positive fragment is not HSCpl .

Note that (c) follows from (a), (b) and the following theorem that holds for any congruence distributive variety of finite signature, and it is interesting in its own right.

Theorem 2.8. *Let $\mathcal{V}$ be a locally finite finitely based congruence distributive variety of finite signature. Then, every subvariety of $\mathcal{V}$ is finitely based if and only if $\mathcal{V}$ has at most countably many subvarieties.*

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Maximality in finite-valued Łukasiewicz logics defined by order filters

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1 Preliminaries and first results

In this talk we consider the logics L_n^i obtained from the $(n+1)$-valued Łukasiewicz logics Ł_{n+1} by taking the order filter generated by i/n as the set of designated elements. The $(n+1)$-valued Łukasiewicz logic can be semantically defined as the matrix logic

$$\text{Ł}_{n+1} = \langle \mathbf{LV_{n+1}}, \{1\} \rangle,$$

where $\mathbf{LV_{n+1}} = (LV_{n+1}, \neg, \rightarrow)$ with $LV_{n+1} = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$, and the operations are defined as follows: for every $x, y \in LV_{n+1}$, $\neg x = 1 - x$ and $x \rightarrow y = \min\{1, 1 - x + y\}$.

Observe that Ł_2 is the usual presentation of classical propositional logic CPL as a matrix logic over the two-element Boolean algebra $\mathbf{B}_2$ with domain $\{0, 1\}$ and signature $\{\neg, \rightarrow\}$. The logics Ł_n can also be presented as Hilbert calculi that are axiomatic extensions of the infinite-valued Łukasiewicz logic Ł_∞.
The following operations can be defined in every algebra $\mathbf{LV_{n+1}}$: $x \otimes y = \neg(x \rightarrow \neg y) = \max\{0, x + y - 1\}$ and $x \oplus y = \neg x \rightarrow y = \min\{1, x + y\}$. For every $n > 1$, $x^n = x \otimes \cdots \otimes x$ (n-times) and $nx = x \oplus \cdots \oplus x$ (n-times).

For $1 \leq i \leq n$ let $F_{i/n} = \{x \in LV_{n+1} \ : \ x \geq i/n\} = \left\{\frac{i}{n}, \ldots, \frac{n-1}{n}, 1\right\}$ be the order filter generated by i/n, and let

$$\mathsf{L}_n^i = \langle \mathbf{LV_{n+1}}, F_{i/n} \rangle$$

be the corresponding matrix logic. From now on, the consequence relation of L_n^i is denoted by $\models_{\mathsf{L}_n^i}$. Observe that $\text{Ł}_{n+1} = \mathsf{L}_n^n$ for every n. In particular, CPL is L_1^1 (that is, Ł_2). If $1 \leq i, m \leq n$, we can also consider the following matrix logic: $\mathsf{L}_m^{i/n} = \langle \mathbf{LV_{m+1}}, F_{i/n} \cap LV_{m+1} \rangle$.
The logic $\mathsf{L}_2^1 = \langle LV_3, \{1, 1/2\} \rangle$ was already known as the 3-valued paraconsistent logic J_3, introduced by da Costa and D'Ottaviano see [4] in order to obtain an example of a paraconsistent logic maximal w.r.t. CPL.

Definition 1. Let L_1 and L_2 be two standard propositional logics defined over the same signature Θ such that L_1 is a proper sublogic of L_2. Then, L_1 is *maximal* w.r.t. L_2 if, for every formula φ over Θ, if $\vdash_{L_2} \varphi$ but $\nvdash_{L_1} \varphi$, then the logic L_1^+ obtained from L_1 by adding φ as a theorem, coincides with L_2.

In order to study maximality among finite-valued Łukasiewicz logics defined by order filters we obtain the following sufficient condition:

Theorem 1. *Let $L_1 = \langle \mathbf{A}_1, F_1 \rangle$ and $L_2 = \langle \mathbf{A}_2, F_2 \rangle$ be two distinct finite matrix logics over a same signature Θ such that $\mathbf{A}_2$ is a subalgebra of $\mathbf{A}_1$ and $F_2 = F_1 \cap A_2$. Assume the following:*

1. *$A_1 = \{0, 1, a_1, \ldots, a_k, a_{k+1}, \ldots, a_n\}$ and $A_2 = \{0, 1, a_1, \ldots, a_k\}$ are finite such that $0 \notin F_1$, $1 \in F_2$ and $\{0, 1\}$ is a subalgebra of $\mathbf{A}_2$.*
2. *There are formulas $\top(p)$ and $\bot(p)$ in $\mathcal{L}(\Theta)$ depending at most on one variable p such that $e(\top(p)) = 1$ and $e(\bot(p)) = 0$, for every evaluation e for L_1.*
3. *For every $k+1 \leq i \leq n$ and $1 \leq j \leq n$ (with $i \neq j$) there exists a formula $\alpha^i_j(p)$ in $\mathcal{L}(\Theta)$ depending at most on one variable p such that, for every evaluation e, $e(\alpha^i_j(p)) = a_j$ if $e(p) = a_i$.*

Then, L_1 is maximal w.r.t. L_2.

We use this result to prove that

Theorem 2. *Let $1 \leq i, m \leq n$. Then L^i_n is maximal w.r.t. $\mathsf{L}^{i/n}_m$ if the following condition holds: there is some prime number p and $k \geq 1$ such that $n = p^k$, and $m = p^{k-1}$.*

Corollary 1. *Let $1 \leq i \leq p$. For every prime number p, L^i_p is maximal w.r.t.* CPL

Notice that the above corollary generalizes the well known result: *L_{p+1} is maximal w.r.t.* CPL *for every prime number p.*

Definition 2. Let L_1 and L_2 be two standard propositional logics defined over the same signature Θ such that L_1 is a proper sublogic of L_2. Then, L_1 is *strongly maximal* w.r.t. L_2 if, for every finitary rule $\varphi_1, \ldots, \varphi_n/\psi$ over Θ, if $\varphi_1, \ldots, \varphi_n \vdash_{L_2} \psi$ but $\varphi_1, \ldots, \varphi_n \nvdash_{L_1} \psi$, then the logic L^*_1 obtained from L_1 by adding $\varphi_1, \ldots, \varphi_n/\psi$ as structural rule, coincides with L_2.

Let i be a strictly positive integer, the *i-explosion rule* is the rule (exp_i) $\dfrac{i(\varphi \wedge \neg\varphi)}{\bot}$.

Lemma 1. *For every $1 \leq i \leq n$, the rule (exp_i) is not valid in L^i_n.*

Corollary 2. *Let $1 \leq i \leq p$. For every prime number p, L^i_p is not strongly maximal w.r.t.* CPL

2 Equivalent systems

Blok and Pigozzi introduce in [3] the notion of equivalent deductive systems in the following sense: Two propositional deductive systems S_1 and S_2 in the same language $\mathcal{L}$ are equivalent iff there are two translations τ_1, τ_2 (finite subsets of $\mathcal{L}$-propositional formulas in one variable) such that:

- $\Gamma \vdash_{S_1} \varphi$ iff $\tau_1(\Gamma) \vdash_{S_2} \tau_1(\varphi)$,
- $\Delta \vdash_{S_2} \psi$ iff $\tau_2(\Delta) \vdash_{S_1} \tau_2(\psi)$,
- $\varphi \dashv\vdash_{S_1} \tau_2(\tau_1(\varphi))$,
- $\psi \dashv\vdash_{S_2} \tau_1(\tau_2(\psi))$.

Theorem 3. *For every $n \geq 2$ and every $1 \leq i \leq n$, L^i_n and L^{n+1} are equivalent deductive systems.*

From the equivalence among L_n^i and Ł$_{n+1}$, we can obtain, by translating the axiomatization of the finite valued Łukasiewicz logic Ł$_{n+1}$, a calculus sound and complete with respect L_n^i that we denote by H_n^i.

Since Ł$_\infty$ is algebraizable and the class MV of all MV-algebras is its equivalent quasivariety semantics, finitary extensions of Ł$_\infty$ are in 1 to 1 correspondence with quasivarieties of MV-algebras. Actually, there is a dual isomorphism from the lattice of all finitary extensions of Ł$_\infty$ and the lattice of all quasivarieties of MV. Moreover, if we restrict this correspondence to varieties of MV we get the dual isomorphism from the lattice of all varieties of MV and the lattice of all axiomatic extensions of Ł$_\infty$. Since Ł$_{n+1} = \mathsf{L}_n^n$ is an axiomatic extension of Ł$_\infty$, Ł$_{n+1}$ is an algebraizable logic with the class $MV_n = \mathcal{Q}(\text{Ł}\mathbf{V}_{n+1})$, the quasivariety generated by Ł$\mathbf{V}_{n+1}$, as its equivalent variety semantics. It follows from the previous theorem that L_n^i, for every $1 \leq i \leq n$, is also algebraizable with the same class of MV_n-algebras as its equivalent variety semantics. Thus, the lattices of all finitary extensions of L_n^i are isomorphic, and in fact, dually isomorphic to the lattice of all subquasivarieties of MV_n, for all $0 < i < n$.

Therefore maximality conditions in the lattice of finitary (axiomatic) extensions correspond to minimality conditions in the lattice of subquasivarieties (subvarieties). Thus, given two finitary extensions L_1 and L_2 of a given logic L_n^i, where K_{L_1} and K_{L_2} are its associated MV_n-quasivarieties, L_1 is strongly maximal with respect L_2 iff K_{L_1} is a minimal subquasivariety of MV_n among those MV_n-quasivarieties properly containing K_{L_2}. Moreover, if L_1 and L_2 are axiomatic extensions of L_n^i, then K_{L_1} and K_{L_2} are indeed MV_n-varieties. In that case, L_1 is maximal with respect L_2 iff K_{L_1} is a minimal subvariety of MV_n among those MV_n-varieties properly containing K_{L_2}.

The lattice of all axiomatic extensions Ł$_\infty$ is fully described also by Komori in [7], thus from the equivalence of Theorem 3, we can obtain the following maximality conditions for all axiomatic extensions of L_n^i.

Theorem 4. *Let $0 < i, m \leq n$ be natural numbers such that $m|n$. If L is an axiomatic extension of L_n^i, then L is maximal with respect to $\mathsf{L}_m^{i/n}$ iff $L = \mathsf{L}_m^{i/n} \cap \mathsf{L}_{p^{k+1}}^{i/n}$ for some prime number p with $p|n$ and a natural $k \geq 0$ such that $p^k|m$ and $p^{k+1} \not| m$.*

As a corollary we obtain that the sufficient condition of Theorem 2 is also necessary.

Corollary 3. *Let $1 \leq i, m \leq n$. Then L_n^i is maximal w.r.t. $\mathsf{L}_m^{i/n}$ if and only if there is some prime number p and $k \geq 1$ such that $n = p^k$, and $m = p^{k-1}$.*

To obtain results on strong maximality we need to study finitary extensions of Ł$_\infty$. The task of fully describing the lattice of all all finitary extensions of Ł$_\infty$, isomorphic to the lattice of all subquasivarieties of MV, turns to be an heroic task since the class of all MV-algebras is Q-universal [1]. For the finite valued case it is much simpler, since MV_n is a locally finite discriminator variety. Any locally finite quasivariety is generated by its critical algebras [5]. Critical MV-algebras were fully described in [6] and using this description we can obtain some results on strong maximality.

First we need to introduce the following matrix logics: For every $1 \leq i, m \leq n$,

$$\bar{\mathsf{L}}_n^i = \langle \mathbf{LV_{n+1}} \times \mathbf{LV_2}, F_{i/n} \times \{1\} \rangle \qquad \bar{\mathsf{L}}_m^{i/n} = \langle \mathbf{LV_{m+1}} \times \mathbf{LV_2}, (F_{i/n} \cap LV_{m+1}) \times \{1\} \rangle$$

Theorem 5. *Let $0 < i \leq n$ be natural numbers, let p be a prime number and let $r = \max\{j \in \mathbb{N} : p^j|n\}$. Then we have: For every j such that $(i-1)p < j \leq ip$, $\mathsf{L}_n^i \cap \bar{\mathsf{L}}_{p^{r+1}}^{j/np}$ is strongly maximal with respect to L_n^i. Moreover, every finitary extension of some L_k^j is strongly maximal with respect L_n^i iff it is one of the preceding types.*

As a particular case we can also prove the following result.

Theorem 6. *Let p be a prime number. Then, for every j such that $0 < j \leq p$:*

- $\bar{\mathsf{L}}_p^j$ *is strongly maximal with respect to* CPL *and it is axiomatized by* H_p^j *plus the j-explosion rule* (exp_j) $j(\varphi \wedge \neg\varphi)/\bot$.
- L_p^j *is strongly maximal w.r.t.* $\bar{\mathsf{L}}_p^j$.

3 Ideal paraconsistent logics

Arieli, Avron and Zamansy introduced in [2] the concept of *ideal paraconsistent logics.*

Definition 3. Let L be a propositional logic defined over a signature Θ (with consecuence relation $\vdash_L$) containing at least a unary connective $\neg$ and a binary connective $\to$ such that:

(i) L is paraconsistent w.r.t. $\neg$, i.e. there are formulas $\varphi, \psi \in \mathcal{L}(\Theta)$ such that $\varphi, \neg\varphi \nvdash_L \psi$; and $\to$ is a deductive implication, i.e. $\Gamma \cup \{\varphi\} \vdash_L \psi$ iff $\Gamma \vdash_L \varphi \to \psi$,.

(ii) There is a presentation of CPL as a matrix logic $L' = \langle \mathbf{A}, \{1\}\rangle$ over the signature Θ such that the domain of $\mathbf{A}$ is $\{0, 1\}$, and $\neg$ and $\to$ are interpreted as the usual 2-valued negation and implication of CPL, respectively, such that L is a sublogic of CPL.

Then, L is said to be an *ideal paraconsistent logic* if it is maximal w.r.t. CPL, and every proper extension of L over Θ is not $\neg$-paraconsistent.

Lemma 2. *Let $0 < i \leq n$. L_n^i is paraconsistent w.r.t. $\neg$ iff $\frac{i}{n} \leq \frac{1}{2}$*

Since for every $0 < i \leq n$, there is a term definable implication $\Rightarrow_n^i$ which is deductive implication next result follows from Theorem 6

Theorem 7. *Let p be a prime number, and let $1 \leq i < p$ such that $i/p \leq 1/2$. Then, L_p^i is a $(p+1)$-valued ideal paraconsistent logic.*[1]

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[1]Strictly speaking, in this claim we implicitly assume that the signature of L_p^i has been changed by adding the definable implication $\Rightarrow_p^i$ as a primitive connective.

On an implication-free reduct of MV_n chains

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Abstract

Let $\mathbf{Ł}_{n+1}$ be the MV-chain on the $n+1$ elements set $Ł_{n+1} = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$ in the algebraic language $\{\to, \neg\}$ [3]. As usual, further operations on $Ł_{n+1}$ are definable by the following stipulations: $1 = x \to x$, $0 = \neg 1$, $x \oplus y = \neg x \to y$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \wedge y = x \odot (x \to y)$, $x \vee y = \neg(\neg x \wedge \neg y)$. Moreover, we will pay special attention to the also definable unary operator ${}^*x = x \odot x$.

In fact, the aim of this paper is to study the $\{^*, \neg, \vee\}$-reducts of the MV-chains $\mathbf{Ł}_{n+1}$, that will be denoted as $\mathbf{Ł}^*_{n+1}$, i.e. the algebra on $Ł_{n+1}$ obtained by replacing the implication operator $\to$ by the unary operation * which represents the square operator ${}^*x = x \odot x$ and which has been recently used in [4] to provide, among other things, an alternative axiomatization for the four-valued matrix logic $J_4 = \langle \mathbf{Ł}_4, \{1/3, 2/3, 1\}\rangle$. In this contribution we make a step further in studying the expressive power of the * operation, in particular we will focus on the question for which natural numbers n the structures $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term-equivalent. In other words, for which n the Łukasiewicz implication $\to$ is definable in $\mathbf{Ł}^*_{n+1}$, or equivalenty, for which n $\mathbf{Ł}^*_{n+1}$ is in fact an MV-algebra. We also show that, in any case, the matrix logics $\langle \mathbf{Ł}^*_{n+1}, F\rangle$, where F is an order filter, are algebraizable. What we present here is a work in progress.

Term-equivalence between $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$

Let X be a subset of $Ł_{n+1}$. We denote by $\langle X\rangle^*$ the subalgebra of $\mathbf{Ł}^*_{n+1}$ generated by X (in the reduced language $\{^*, \neg, \vee\}$). For $n \geq 1$ define recursively $(^*)^n x$ as follows: $(^*)^1 x = {}^*x$, and $(^*)^{i+1}x = {}^*((^*)^i x)$, for $i \geq 1$.

A nice feature of the $\mathbf{Ł}^*_{n+1}$ algebras is that we can always define terms characterising the principal order filters $F_a = \{b \in Ł_{n+1} \mid a \leq b\}$, for every $a \in Ł_{n+1}$.

Proposition 1. *For each $a \in L_{n+1}$, the unary operation Δ_a defined as*

$$\Delta_a(x) = \begin{cases} 1 & \text{if } x \in F_a \\ 0 & \text{otherwise.} \end{cases}$$

is definable in $\mathbf{Ł}^*_{n+1}$. As a consequence, for every $a \in Ł_{n+1}$, the operation χ_a that corresponds to the characteristic function of a (i.e. $\chi_a(x) = 1$ if $x = a$ and $\chi_a(x) = 0$ otherwise) is definable as well.

Proof. The case $a = 1$ corresponds to the Monteiro-Baaz Delta operator and, as is well-known, it can be defined as $\Delta_1(x) = (^*)^n x$. For $a = 0$ define $\Delta_a(x) = \Delta_1(x) \vee \neg\Delta_1(x)$; then $\Delta_a(x) = 1$ for every x. Now, assume $0 < a = i/n < 1$. It is not difficult to show that one can always find a sequence of terms (operations) $t_1(x), \ldots, t_m(x)$ over $\{*, \neg\}$ such that $t_1(t_2(\ldots(t_m(x))\ldots)) = 1$

if $x \in F_a$ while $t_1(t_2(\ldots(t_m(x))\ldots)) < 1$ otherwise. Then $\Delta_a(x) = \Delta_1(t_1(t_2(\ldots(t_m(x))\ldots)))$ for every x.

As for the operations χ_a, define $\chi_1 = \Delta_1$, $\chi_0 = \neg\Delta_{1/n}$, and if $0 < a < 1$, then define $\chi_a = \Delta_a \wedge \neg\Delta_{a-1/n}$. □

It is now almost immediate to check that the following implication-like operation is definable in every $\mathbf{Ł}^*_{n+1}$: $x \Rightarrow y = 1$ if $x \leq y$ and 0 otherwise. Indeed, $\Rightarrow$ can be defined as

$$x \Rightarrow y = \bigvee_{0\leq i\leq j\leq n} (\chi_{i/n}(x) \wedge \chi_{j/n}(y)).$$

Actually, one can also define Gödel implication on $\mathbf{Ł}^*_{n+1}$ by putting $x \Rightarrow_G y = (x \Rightarrow y) \vee y$.

On the other hand, it readily follows from Proposition 1 that all the $\mathbf{Ł}^*_{n+1}$ algebras are simple. Indeed, if $a > b \in \text{Ł}_{n+1}$ would be congruent, then $\Delta_a(a) = 1$ and $\Delta_a(b) = 0$ should be so. Recall that an algebra is called *strictly simple* if it is simple and does not contain proper subalgebras. It is clear then that in the case of $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ algebras, they are strictly simple if $\{0, 1\}$ is their only proper subalgebra.

Remark 2. It is well-known that $\mathbf{Ł}_{n+1}$ is strictly simple iff n is prime. Note that, for every n, if $\mathbf{B} = (B, \neg, \to)$ is an MV-subalgebra of $\mathbf{Ł}_{n+1}$, then $\mathbf{B}^* = (B, \vee, \neg, *)$ is a subalgebra of $\mathbf{Ł}^*_{n+1}$ as well. Thus, if $\mathbf{Ł}_{n+1}$ is not strictly simple, then $\mathbf{Ł}^*_{n+1}$ is not strictly simple as well. Therefore, if n is not prime, $\mathbf{Ł}^*_{n+1}$ is not strictly simple. However, in contrast with the case of $\mathbf{Ł}_{n+1}$, n being prime is not a sufficient condition for $\mathbf{Ł}^*_{n+1}$ being strictly simple. In Lemma 7 below we will provide some examples of prime n for which $\mathbf{Ł}^*_{n+1}$ is not strictly simple, in view of Theorem 6.

Lemma 3. *$\mathbf{Ł}^*_{n+1}$ is strictly simple iff $\langle (n-1)/n\rangle^* = \mathbf{Ł}^*_{n+1}$.*

Proof. The 'only if' direction is trivial. In order to prove the converse, assume that $\langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$ for $a_1 = (n-1)/n$. For $i \geq 1$ let $a_{i+1} = t_i(a_i)$ such that $t_i(x) = {}^*x$ if $x > 1/2$, and $t_i(x) = \neg x$ otherwise. Since $\mathbf{Ł}^*_{n+1}$ is finite, there is $1 \leq i < j$ such that $a_j = a_i$ and so $A_1 := \{a_i \mid i \geq 1\} = \{a_i \mid 1 \leq i \leq k\}$ for some k such that $a_i \neq a_j$ if $1 \leq i, j \leq k$. Let $A = A_1 \cup A_2 \cup \{0, 1\}$ where $A_2 = \{\neg a \mid a \in A_1\}$. Since ${}^*1 = 1$ and ${}^*x = 0$ if $x \leq 1/2$, A is the domain of a subalgebra $\mathbf{A}$ of $\mathbf{Ł}^*_{n+1}$ over $\{{}^*, \neg, \vee\}$ such that $a_1 \in A$, hence $\langle a_1\rangle^* \subseteq \mathbf{A}$. But $\mathbf{A} \subseteq \langle a_1\rangle^*$, by construction. Therefore $\mathbf{A} = \langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$.

Fact: Under the current hypothesis (namely, $\langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$): if n is even then $n = 2$ or $n = 4$. Indeed, suppose that $\langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$ and n is even. If $n = 2$ or $n = 4$ then clearly $\mathbf{Ł}^*_{n+1}$ is strictly simple. Now, assume $n > 4$. Observe that: (1) for any $a \in \mathbf{Ł}^*_{n+1} \setminus \{0, 1\}$, ${}^*a = i/n$ such that i is even; and (2) if $i < n$ is even then $\neg(i/n) = (n-i)/n$ such that $n - i$ is even. That being so, if $i/n \in (A_1 \cup A_2) \setminus \{a_1, \neg a_1\}$ (recall the process described above) then i is even. But then, for instance, $3/n \notin \mathbf{A} = \langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$, a contradiction. This proves the Fact.

From the **Fact**, assume now that n is odd, and let $a = ((n+1)/2)/n$ and $b = ((n-1)/2)/n$. Since $\neg a = b$, $\neg b = a$ and $a, b \in A$ then, by construction of A, there is $1 \leq i \leq k$ such that either $a = a_i$ or $b = a_i$. If $a = a_i$ then $a_{i+1} = {}^*a = 1/n$ and so $a_{i+2} = \neg a_{i+1} = \neg 1/n = (n-1)/n = a_1$. Analogously it can be proven that, if $b = a_i$ then $a_1 = a_j$ for some $j > i$. This shows that $A_1 = \{a_1, \ldots, a_k\}$ is such that $a_{k+1} = a_1$ (hence $a_k = 1/n$). Now, let $c \in \mathbf{Ł}^*_{n+1} \setminus \{0, 1\}$ such that $c \neq a_1$. If $c \in A_1$ then the process of generation of A from c will produce the same set A_1 and so $\mathbf{A} = \mathbf{Ł}^*_{n+1}$, showing that $\langle c\rangle = \mathbf{Ł}^*_{n+1}$. Otherwise, if $c \in A_2$ then $\neg c \in A_1$ and, by the same argument as above, it follows that $\langle c\rangle = \mathbf{Ł}^*_{n+1}$. This shows that $\mathbf{Ł}^*_{n+1}$ is strictly simple. □

Lemma 4. *If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then $\mathbf{Ł}^*_{n+1}$ is strictly simple.*

Proof. If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then $\odot$ is definable in $\mathbf{Ł}^*_{n+1}$, and hence $\langle(n-1)/n\rangle^* = \mathbf{Ł}^*_{n+1}$. Indeed, we can obtain $(n-i-1)/n = ((n-1)/n) \odot ((n-i)/n)$ for $i = 1, \ldots, n-1$, and $1 = \neg 0$. By Lemma 3 it follows that $\mathbf{Ł}^*_{n+1}$ is strictly simple. □

Corollary 5. *If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then n is prime.*

Proof. If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then $\mathbf{Ł}^*_{n+1}$ is strictly simple, by Lemma 4. By Remark 2 it follows that n must be prime. □

Theorem 6. *$\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ iff $\mathbf{Ł}^*_{n+1}$ is strictly simple.*

Proof. The 'only if' part is Lemma 4. For the 'if' part, since $\mathbf{Ł}^*_{n+1}$ is strictly simple then, for each $a, b \in Ł_{n+1}$ where $a \notin \{0,1\}$ there is a definable term $\mathbf{t}_{a,b}(x)$ such that $\mathbf{t}_{a,b}(a) = b$. Otherwise, if for some $a \notin \{0,1\}$ and $b \in Ł_{n+1}$ there is no such term then $\mathbf{A} = \langle a\rangle^*$ would be a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ (since $b \notin \mathbf{A}$) different from $\{0,1\}$, a contradiction. By Proposition 1 the operations $\chi_a(x)$ are definable for each $a \in Ł_{n+1}$, then in $\mathbf{Ł}^*_{n+1}$ we can define Łukasiewicz implication $\to$ as follows:

$$x \to y = (x \Rightarrow y) \vee \left(\bigvee_{n>i>j\geq 0} \chi_{i/n}(x) \wedge \chi_{j/n}(y) \wedge \mathbf{t}_{i/n,a_{ij}}(x) \right) \vee \left(\bigvee_{n>j\geq 0} \chi_1(x) \wedge \chi_{j/n}(y) \wedge y \right)$$

where $a_{ij} = 1 - i/n + j/n$. □

We have seen that n being prime is a necessary condition for $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ being term-equivalent. But this is not a sufficient condition: in fact, there are prime numbers n for which $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are not term-equivalent.

Lemma 7. *If n is a prime Fermat number greater than 5 then $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are not term-equivalent.*

Proof. Recall that a Fermat number is of the form $2^{2^k} + 1$, with k being a natural number. We are going to prove that if n is a prime Fermat number and $a_1 = (n-1)/n$, then $\langle a_1\rangle^*$ is a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ (recall Theorem 6 and Lemma 3). Thus, let $n > 5$ be a prime Fermat number, that is, a prime number of the form $n = 2^m + 1$ with $m = 2^k$ and $k > 1$. The $(m-1)$-times iterations of $*$ applied to a_1 produce $((n+1)/2)/n$, that is: $(^*)^{m-1}(a_1) = ((n+1)/2)/n$. Since $^*(((n+1)/2)/n) = 1/n$, the constructive procedure for generating the algebra $\langle a_1\rangle^*$ described in the proof of Lemma 3 shows that $\langle a_1\rangle^* = \mathbf{A}$ has $2m+2$ elements: m elements in A_1, plus m elements in A_2 corresponding to their negations, plus 0 and 1. Since $2m+2 < 2^m+1 = n$ as $n > 5$, $\langle a_1\rangle^*$ is properly contained in $Ł_{n+1}$, and it is different from $\{0,1\}$. □

The first Fermat prime number greater than 5 is $n = 17$. It is easy to see that

$$\langle 16/17\rangle^* = \{0, 1/17, 2/17, 4/17, 8/17, 9/17, 13/17, 15/17, 16/17, 1\}.$$

Actually, we do not have a full characterisation of those prime numbers n for which $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term-equivalent. But computational results show that for prime numbers until 8000, about 60% of the cases yield term-equivalence.

Algebraizability of $\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$

Given the algebra $\mathbf{Ł}^*_{n+1}$, it is possible to consider, for every $1 \leq i \leq n$, the matrix logic $Ł^*_{i,n+1} = \langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$. In this section we will shown that all the $Ł^*_{i,n+1}$ are algebraizable in the sense of Blok-Pigozzi [1], and the quasivarieties associated to $Ł^*_{i,n+1}$ and $Ł^*_{j,n+1}$ are the same, for every i, j.

Observe that the operation $x \approx y = 1$ if $x = y$ and $x \approx y = 0$ otherwise is definable in $\mathbf{Ł}^*_{n+1}$. Indeed, it can be defined as $x \approx y = (x \Rightarrow y) \wedge (y \Rightarrow x)$. Also observe that $x \approx y = \Delta_1((x \Rightarrow_G y) \wedge (y \Rightarrow_G x))$ as well.

In order to prove the main result of this section, we state the following:

Lemma 8. *For every n, the logic $Ł^*_{n+1} := Ł^*_{n,n+1} = \langle \mathbf{Ł}^*_{n+1}, \{1\}\rangle$ is algebraizable.*

Proof. It is immediate to see that the set of formulas $\Delta(p, q) = \{p \approx q\}$ and the set of pairs of formulas $E(p, q) = \{\langle p, \Delta_0(p)\rangle\}$ satisfy the requirements of algebraizability. ☐

Blok and Pigozzi [2] introduce the following notion of equivalent deductive systems. Two propositional deductive systems S_1 and S_2 in the same language are *equivalent* if there are translations $\tau_i : S_i \to S_j$ for $i \neq j$ such that: $\Gamma \vdash_{S_i} \varphi$ iff $\tau_i(\Gamma) \vdash_{S_j} \tau_i(\varphi)$, and $\varphi \dashv\vdash_{S_i} \tau_j(\tau_i(\varphi))$. From very general results in [2] it follows that two equivalent logic systems are indistinguishable from the point of view of algebra, namely: if one of the systems is algebraizable then the other will be also algebraizable w.r.t. the same quasivariety. This will be applied to $Ł^*_{i,n+1}$.

Lemma 9. *The logics $Ł^*_{n+1}$ and $Ł^*_{i,n+1}$ are equivalent, for every n and for every $1 \leq i \leq n-1$.*

Proof. It is enough to consider the translation mappings $\tau_1 : Ł^*_{n+1} \to Ł^*_{i,n+1}$, $\tau_1(\varphi) = \Delta_1(\varphi)$, and $\tau_{i,2} : Ł^*_{i,n+1} \to Ł^*_{n+1}$, $\tau_{i,2}(\varphi) = \Delta_{i/n}(\varphi)$. ☐

Finally, as a direct consequence of Lemma 8, Lemma 9 and the observations above, we can prove the following result.

Theorem 10. *For every n and for every $1 \leq i \leq n$, the logic $Ł^*_{i,n+1}$ is algebraizable.*

As an immediate consequence of Theorem 10, for each logic $Ł^*_{i,n+1}$ there is a quasivariety $\mathcal{Q}(i, n)$ which is its equivalent algebraic semantics. Moreover, by Lemma 9 and by Blok and Pigozzi's results, $\mathcal{Q}(i, n)$ and $\mathcal{Q}(j, n)$ coincide, for every i, j. The question of axiomatising $\mathcal{Q}(i, n)$ is left for future work.

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A fuzzy-paraconsistent version of basic hybrid logic

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Inconsistencies are no longer seen only as anomalies: it is a fact that collecting data from different sources introduces them quite frequently. This may happen in systems which are safety critical, such as health systems, aviation systems and many others. As a means of increasing the reliability of those systems, one resorts to paraconsistent reasoning in order to suitably address inconsistencies. Paraconsistent logics allow the coexistence of contradictory information without the collapse of the whole system. This field has been driven not only by theoretical interest, but also by genuine problems in different scientific domains, such as Computer Science, Medicine or Robotics.

There are several results connecting paraconsistency with modal logics. Quasi-hybrid (QH) logic, as introduced in [4], combines basic hybrid logic – an extension of modal logic able to refer to specific states through the introduction of new formulas, called nominals, for naming states and a satisfaction operator @ such that $@_i\phi$ is true if and only if at the state whose name is i, ϕ is true – with paraconsistency – where a formula and its negation are allowed to be simultaneously true. QH logic is able to accommodate local inconsistencies by splitting the usual hybrid valuation into two, V^+ and V^-, such that positive literals, $@_ip$, are evaluated resorting to the former, and negative literals, $@_i\neg p$, to the latter.

A model is described as the set of hybrid literals that are true in the model, and it is possible to construct models for a set of formulas that contain inconsistencies. For the models with minimum amount of information still capable of satisfying the database given, it is introduced a measure of inconsistency that helps comparing between models, and then choosing the *least inconsistent*. Another measure of inconsistency assigns weights to variables so that one can take into account in which variables it is more important to keep consistency.

There is a sound and complete tableau system for QH logic [3], obtained by combining a tableau system for Quasi-classical logic with one for Hybrid logic, [5] and [2, 1] respectively.

However, in this setting, as in classical logic, valuations are thought of as giving the values 0 or 1 all the time. When dealing with information that we gather from different sources, assigning only those values is not enough to express the certainty of such affirmations. How many times have we asked "Are you 100% sure?" and got as answer "No, but I am 90%."? With this in mind, it seemed only natural to redesign the aforementioned Quasi-hybrid logic into a logic where valuations behave in a fuzzy manner, and where both inconsistent and incomplete formulas are allowed.

The *"Fuzzified"* version of Quasi-hybrid logic proposed here also uses two valuations, but now, in addition to the propositional variable being evaluated, valuations have embedded a local perspective, as follows: $V^* : \mathrm{Prop} \times \mathrm{W} \to [0,1]$, $* \in \{+,-\}$. So, we can define thresholds $a, b \in [0,1]$ such that $@_i p$ is true if $V^+(p,w) > a$, where w is the world named by the nominal i and it is false if $V^-(p,w) > b$. V^+ and V^- cannot be extended to formulas as usual. Observe that a and b do not need to be equal, that one could even set different thresholds for each propositional variable and world, and that one could eventually consider the relation $\geq$. In some sense, $V^+(p,w)$ can be regarded as evidence that supports p at w and $V^-(p,w)$ as evidence that denies p at w. The higher a and b, the more we demand on the certainty of the information.

In this new logic, reductio ad absurdum and disjunctive syllogism are dropped, but the rules of weakening, double negation elimination, disjunction introduction and transitivity still hold.

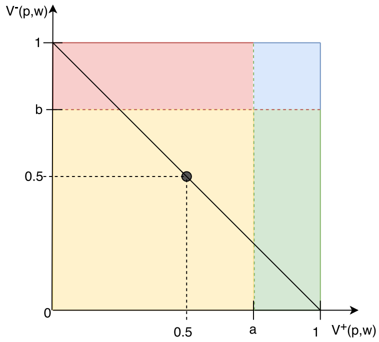
We can analyze the connection between $V^+(p,w)$ and $V^-(p,w)$ from two perspectives: first in terms of the quality of the information, and second in terms of quantity. For the former we have that:

- if $V^+(p,w) > a$ and $V^-(p,w) \leq b$, then p is true at w;
- if $V^+(p,w) \leq a$ and $V^-(p,w) > b$, then p is false at w;
- if $V^+(p,w) > a$ and $V^-(p,w) > b$, then p is both true and false at w, thus p is said to be *inconsistent* at w;
- if $V^+(p,w) \leq a$ and $V^-(p,w) \leq b$, then p is neither true nor false at w, thus p is said to be *incomplete* at w.

For the second perspective,

- if $V^+(p,w) + V^-(p,w) = 1$, then we have the expected amount of information;
- if $V^+(p,w) + V^-(p,w) > 1$, then we have more information than expected about p at w, and we call it *overinformation*;
- if $V^+(p,w) + V^-(p,w) < 1$, then we have less information than expected about p at w, and we call it *underinformation*.

The figure below sketches the two perspectives overlapping, with $a + b > 1$.



In the green area p is true at w, in the red area $\neg p$ is true at w, in the blue area, both p and $\neg p$ are true at w, and finally in the yellow area neither p nor $\neg p$ are true at w. Above the line $V^+(p,w) + V^-(p,w) = 1$ we have overinformation, and below it we have underinformation.

We can derive very interesting – and sometimes very intriguing – conclusions when using this logic. In the previous example, there are regions which correspond to full information (even overinformation) and yet neither p nor $\neg p$ are true; that is because we are demanding a lot with the thresholds set as they were.

Given a set of formulas Δ, we can always find a model such that all formulas in Δ are satisfied. Consider the following example: let $\Delta = \{@_i\Diamond\neg p, @_i\Box p\}$ and set $@_i p$ to be true if $V^+(p,w) > 0.6$ and false if $V^-(p,w) > 0.3$. A model for Δ could be $M = \langle W, R, N, V^+, V^-\rangle$ where $W = \{w\}, R = \{(w,w)\}$, N a function that assigns nominals to worlds, in this case $N : \{i\} \to \{w\}$, $V^+(p,w) = 0.61$ and $V^-(p,w) = 0.31$. Observe that, however, it is not always the case that we can find a model where $V^+(p,w) + V^-(p,w) = 1$ for all $p \in \mathrm{Prop}, w \in W$.

We can describe several measures of *inconsistency* for a model $\mathcal{M}$, either absolute or relative ones, from the combination of the perspectives enumerated above. One that we would like to address, regarding quantity of information, is the following:

Define $A(\mathcal{M}) = \{(p,w) \mid p \in \mathrm{Prop},\ w \in W,\ V^+(p,w) + V^-(p,w) > 1\}$ and $B(\mathcal{M}) = \{(p,w) \mid p \in \mathrm{Prop},\ w \in W,\ V^+(p,w) + V^-(p,w) < 1\}$,

$$\mathrm{MOver}(\mathcal{M}) = \sum_{A(\mathcal{M})} [(V^+(p,w) + V^-(p,w)) - 1]$$

and

$$\mathrm{MUnder}(\mathcal{M}) = \sum_{B(\mathcal{M})} [1 - (V^+(p,w) + V^-(p,w))].$$

The total measure (a relative one) would be given as follows:

$$\mathrm{MQuant}(\mathcal{M}) = \frac{\mathrm{MOver}(\mathcal{M}) + \mathrm{MUnder}(\mathcal{M})}{|\mathrm{Prop}| \times |W|}.$$

From the point of view of the quality of information, consider the following:

$$\mathrm{MIncons}(\mathcal{M}) = |\{(p,w) \mid V^+(p,w) > a_{p,w}\ \&\ V^-(p,w) > b_{p,w}\}|$$

and

$$\mathrm{MIncomp}(\mathcal{M}) = |\{(p,w) \mid V^{+}(p,w) \leq a_{p,w} \;\&\; V^{-}(p,w) \leq b_{p,w}\}|.$$

where $a_{p,w}$ and $b_{p,w}$ are the thresholds for which $@_i p$ and $@_i \neg p$ hold, respectively (w is the world named by i).

Thus for this perspective, we propose the following relative measure:

$$\mathrm{MQual}(\mathcal{M}) = \frac{\mathrm{MIncons}(\mathcal{M}) + \mathrm{MIncomp}(\mathcal{M})}{|\mathrm{Prop}| \times |W|}.$$

Then we can combine $\mathrm{MQuant}(\mathcal{M})$ and $\mathrm{MQual}(\mathcal{M})$.

We would like to point out that both $\mathrm{MOver}(\mathcal{M})$, $\mathrm{MUnder}(\mathcal{M})$, $|A(\mathcal{M})|$, $|B(\mathcal{M})|$, $\mathrm{MIncons}(\mathcal{M})$ and $\mathrm{MIncomp}(\mathcal{M})$ could be considered by themselves as absolute measures.

For the example given, we get the next results:

$$\begin{array}{l|l} \mathrm{MOver}(\mathcal{M}) = 0 & \mathrm{MIncons}(\mathcal{M}) = 1 \\ \mathrm{MUnder}(\mathcal{M}) = 1 - 0.92 = 0.08 & \mathrm{MIncomp}(\mathcal{M}) = 0 \\ \mathrm{MQuant}(\mathcal{M}) = \frac{0.08}{1} = 0.08 & \mathrm{MQual}(\mathcal{M}) = \frac{1}{1} = 1 \end{array}$$

Conclusion. This work relates the subjects of paraconsistency, fuzzy valuations and hybrid logic. We would like to continue exploring measures of inconsistency in this scenario, as well as to study some applications in which this approach might be useful, namely when valuations V^{+} and V^{-} correspond to information obtained by the use of complementar sensors, in which case getting values such that $V^{+} + V^{-} \neq 1$ reveals that at least one of them is malfunctioning.

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Functionality Property in Partial Fuzzy Logic

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Abstract

We study a functionality property of partially-defined fuzzy relations, i.e., fuzzy relations with membership functions that are not necessarily defined everywhere. We handle them in a partial fuzzy logic. We analyze some selected known results for functional fuzzy relations and we extend them suitability to functional partial fuzzy relations.

1 Introduction

Functionality property is one of the crucial properties of fuzzy set theory from the theoretical as well as practical point of view. In [5] a functionality property has been studied for standard fuzzy relations and we switch our background logic to Partial Fuzzy Logic (PFL) [3]. This logic is designed to handle objects that can have undefined membership degrees. We will provide some initial results on the preservation of partial graded functionality under partial fuzzy set and relational operations.

2 Partial fuzzy propositional logic

Partial fuzzy logic proposed in [3] is based on any Δ-core [7] fuzzy logic L and defined as follows (for details see [3]):

- The *language* (or signature) $\mathscr{S}^*$ of L^* extends the language $\mathscr{S}$ of L by the truth constant $*$ (representing the undefined truth degree of propositions), the unary connective ! (for the crisp modality "is defined"), the unary Bochvar-style connective u_{B} for each unary $u \in \mathscr{S}$, the binary Bochvar-style connectives c_{B} for each binary $c \in \mathscr{S}$ and the binary connective $\wedge_{\mathrm{K}}$ (for Kleene-style min-conjunction).

- Intended *algebras* (of truth values) for L^* are defined by expanding the algebras for L by a dummy element $*$ (to be assigned to propositions with undefined truth). In the intended L^*-algebra $\boldsymbol{L}_* = \boldsymbol{L} \cup \{*\}$ (where $\boldsymbol{L}$ is an L-algebra), the connectives of L^* are interpreted as described by the following truth tables, for all unary connectives $u \in \mathscr{S}$, binary connectives $c \in \mathscr{S}$ (and similarly for higher arities), $\alpha, \beta \in \boldsymbol{L}$ and $\gamma, \delta \in \boldsymbol{L} \setminus \{0\}$:

$$
\begin{array}{c|c}
 & ! \\ \hline
\alpha & 1 \\
* & 0
\end{array}
\qquad
\begin{array}{c|c}
 & u_{\mathrm{B}} \\ \hline
\alpha & u\alpha \\
* & *
\end{array}
\qquad
\begin{array}{c|cc}
c_{\mathrm{B}} & \beta & * \\ \hline
\alpha & \alpha\, c\, \beta & * \\
* & * & *
\end{array}
\qquad
\begin{array}{c|ccc}
\wedge_{\mathrm{K}} & 0 & \delta & * \\ \hline
0 & 0 & 0 & 0 \\
\gamma & 0 & \gamma \wedge \delta & * \\
* & 0 & * & *
\end{array}
\tag{1}
$$

- *Tautologies* of L^* are defined as formulae that are evaluated to 1 under all evaluations in all intended L^*-algebras. *Entailment* in L^* is defined as transmission of the value 1 under all evaluations in all intended L^*-algebras. As usual, we write $\models \varphi$ to indicate the tautologicity of φ in L^* and $\Gamma \models \varphi$ to denote the fact that the set Γ of formulae entails the formula φ in L^*.

Since in this paper we only deal with the semantics of fuzzy partial logic and fuzzy partial set theory, we leave the axiomatic system for L^* aside.

The connectives of $\mathscr{S}^*$ make a broad class of derived connectives available in L^*(for more details see [3]). This includes several useful families of connectives well-known from three-valued logic (see, e.g., [4]), from which we will use the following two:

- The *Bochvar-style* connectives $c_B \in \{\wedge_B, \vee_B, \&_B\}$, which treat $*$ as the annihilator. Recall that in L^*, the connectives of the original language $\mathscr{S}$ of the underlying fuzzy logic L are actually interpreted Bochvar-style: see the truth tables (1) above.

- The *Sobociński-style* connectives $c_S \in \{\wedge_S, \vee_S, \&_S\}$, which treat $*$ as the neutral element.

Moreover, the following useful unary and binary connectives are L^*-definable:

$$\begin{array}{c|ccc} x & ?x & \downarrow x & \uparrow x \\ \hline \alpha & 0 & \alpha & \alpha \\ * & 1 & 0 & 1 \end{array} \qquad \begin{array}{c|cc} \rightarrow_* & \beta & * \\ \hline \alpha & \alpha \rightarrow \beta & 0 \\ * & 1 & 1 \end{array} \qquad \begin{array}{c|cc} \rightarrow^* & \beta & * \\ \hline \alpha & \alpha \rightarrow \beta & 1 \\ * & 0 & 1 \end{array} \tag{2}$$

for $\alpha, \beta \neq *$. For more details on fuzzy partial propositional logic (including examples of valid laws and several metamathematical results) see [3].

A predicate variant $L\forall^*$ of L^* can be defined in a manner analogous to other fuzzy first-order logics (cf., e.g., [1]). Like in propositional L^*, the (standardly defined) first-order formulae of $L\forall^*$ are evaluated in L^*-algebras of truth values. The interpretation of a unary (and analogously higher-arity) predicate symbol P in a given model $\mathbf{M}$ is thus a *total* function $P_{\mathbf{M}} \colon D_{\mathbf{M}} \to \boldsymbol{L}_*$, where $D_{\mathbf{M}}$ is the domain of $\mathbf{M}$ and $\boldsymbol{L}_*$ is an intended L^*-algebra of truth values.

The Tarski conditions for terms, atomic formulae, and propositional connectives are defined as usual; because of space limitations, we omit them here and refer the reader to [1, Sect. 5]. Let the truth value (in $\boldsymbol{L}_*$) of the formula φ in a model $\mathbf{M}$ under an evaluation e of individual variables be denoted by $\|\varphi\|^{\mathbf{M}}_e$.

The primitive quantifiers $\forall_B$, $\exists_B$ of $L\forall^*$ are interpreted Bochvar-style, yielding $*$ whenever there is an undefined instance of the quantified formula. $\forall_B$, $\exists_B$ are of limited utility but they are sufficient for the definability of further useful quantifiers in $L\forall^*$ by means of the connectives of L^*. One of them is the Sobociński-style quantifiers with the following Tarski conditions:

$$\|(\forall_S x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for all } a \in D_{\mathbf{M}} \\ \inf_{a\in D_{\mathbf{M}}} \|\uparrow\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} & \text{otherwise} \end{cases} \tag{3}$$

$$\|(\exists_S x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for all } a \in D_{\mathbf{M}} \\ \sup_{a\in D_{\mathbf{M}}} \|\downarrow\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} & \text{otherwise} \end{cases} \tag{4}$$

In a standard manner we introduce partial fuzzy logic of a higher order, see [2].

3 Partial fuzzy relational operations

Due to a various treatments of undefined values we have the following two main options to define relational operations:

- *Bochvar intersection*: $(R \sqcap_{\mathrm{B}} S)(x,y) \equiv_{\mathrm{df}} R(x,y) \wedge_{\mathrm{B}} S(x,y)$,
- *Sobociński intersection*: $(R \sqcap_{\mathrm{S}} S)(x,y) \equiv_{\mathrm{df}} R(x,y) \wedge_{\mathrm{S}} S(x,y)$,
- *Bochvar strong-intersection*: $(R \cap_{\mathrm{B}} S)(x,y) \equiv_{\mathrm{df}} R(x,y) \,\&_{\mathrm{B}}\, S(x,y)$,
- *Sobociński strong-intersection*: $(R \cap_{\mathrm{S}} S)(x,y) \equiv_{\mathrm{df}} R(x,y) \,\&_{\mathrm{S}}\, S(x,y)$.

The following relational composition sup-T of partial fuzzy relations is intended to be computed in a standard way on the intersection of the domains of partial fuzzy relations and otherwise it should remain undefined. It leads to the following definition by means of Bochvar and Sobociński connectives and quantifiers:

- *Sobociński–Bochvar sup-T composition*

$$(R \circ_{\mathrm{SB}} S)(x,y) \equiv_{\mathrm{df}} (\exists_{\mathrm{S}} z)\big(R(x,z) \,\&_{\mathrm{B}}\, S(z,y)\big),$$

Furthermore, let us define the following graded and crisp properties of fuzzy relations:

- *Sobociński–Bochvar subsethood*: $R \subseteq_{\mathrm{SB}} S \equiv_{\mathrm{df}} (\forall_{\mathrm{S}} x)(\forall_{\mathrm{S}} y)\big(R(x,y) \to_{\mathrm{B}} S(x,y)\big)$,
- *Sobociński–Sobociński subsethood*: $R \subseteq_{\mathrm{SS}} S \equiv_{\mathrm{df}} (\forall_{\mathrm{S}} x)(\forall_{\mathrm{S}} y)\big(R(x,y) \to_{\mathrm{S}} S(x,y)\big)$,
- *Totalness*: $\mathrm{Tot}(R) \equiv_{\mathrm{df}} (\forall_{\mathrm{B}} x)(\forall_{\mathrm{B}} y)!R(x,y)$.

4 Functionality property of partial fuzzy relations

In our approach, functionality property is a direct generalization of the classical property that specifies functions out of relations. In the fuzzy community the functionality property has been studied by many authors and it is also known as the unique mapping [6]. We will propose the notion of functionality for partial fuzzy relations in agreement with our intuitive expectations. Further, we will study its properties w.r.t. fuzzy set operations and a relational composition.

Definition 1. *The* functionality *property of R w.r.t. $\approx_1, \approx_2$ is defined as*

$$\mathrm{Func}_{\approx_1,\approx_2}(R) \equiv_{\mathrm{df}} (\forall_{\mathrm{S}} x)(\forall_{\mathrm{S}} x')(\forall_{\mathrm{S}} y)(\forall_{\mathrm{S}} y')(x \approx_1 x' \,\&_{\mathrm{B}}\, R(x,y) \,\&_{\mathrm{B}}\, R(x',y') \to_{\mathrm{B}} y \approx_2 y').$$

Let us give some examples of tautologies of $\mathrm{L}\forall^*$ for relational operations:

$$\mathrm{Tot}(\approx_1), \mathrm{Tot}(\approx_2) \models_{\mathrm{L}\forall^*} \mathrm{Func}_{\approx_1,\approx_2}(F) \wedge_{\mathrm{B}} \mathrm{Func}_{\approx_1,\approx_2}(G) \to^* \mathrm{Func}_{\approx_1,\approx_2}(F \sqcap_{\mathrm{B}} G),$$
$$\mathrm{Tot}(\approx_1), \mathrm{Tot}(\approx_2) \models_{\mathrm{L}\forall^*} \mathrm{Func}_{\approx_1,\approx_2}(F) \,\&_{\mathrm{B}}\, \mathrm{Func}_{\approx_1,\approx_2}(G) \to^* \mathrm{Func}_{(\approx_1)^2,(\approx_2)^2}(F \cap_{\mathrm{B}} G),$$

where $x \approx_i^2 x' \equiv_{\mathrm{df}} (x \approx_i x') \,\&_{\mathrm{B}}\, (x \approx_i x')$, $i = 1, 2$.

The situation is analogous with Sobociński operations. The resulting domain of Sobociński operations is the union of domains of the input relations. Therefore, the following formulas work for both orderings induced by $\to_*$ and $\to^*$. Here, we will present tautologies only with $\to^*$:

$$\mathrm{Tot}(\approx_1), \mathrm{Tot}(\approx_2) \models_{\mathrm{L}\forall^*} \mathrm{Func}_{\approx_1,\approx_2}(F) \wedge_{\mathrm{S}} \mathrm{Func}_{\approx_1,\approx_2}(G) \to^* \mathrm{Func}_{\approx_1,\approx_2}(F \sqcap_{\mathrm{S}} G),$$
$$\mathrm{Tot}(\approx_1), \mathrm{Tot}(\approx_2) \models_{\mathrm{L}\forall^*} \mathrm{Func}_{\approx_1,\approx_2}(F) \,\&_{\mathrm{S}}\, \mathrm{Func}_{\approx_1,\approx_2}(G) \to^* \mathrm{Func}_{(\approx_1)^2,(\approx_2)^2}(F \cap_{\mathrm{S}} G).$$

Moreover, the following formulas hold for the relational composition of partial fuzzy relations:

$$\mathrm{Tot}(\approx_1), \mathrm{Tot}(\approx_2), \mathrm{Tot}(\approx_3) \models_{\mathrm{L}\forall^*} \mathrm{Func}_{\approx_1,\approx_2}(F) \mathbin{\&_{\mathrm{B}}} \mathrm{Func}_{\approx_2,\approx_3}(G) \to^* \mathrm{Func}_{\approx_1,\approx_3}(F \circ_{\mathrm{SB}} G),$$
$$\mathrm{Tot}(\approx_1), \mathrm{Tot}(\approx_2), \mathrm{Tot}(\approx_3) \models_{\mathrm{L}\forall^*} \mathrm{Func}_{\approx_1,\approx_2}(F) \mathbin{\&_{\mathrm{S}}} \mathrm{Func}_{\approx_2,\approx_3}(G) \to_* \mathrm{Func}_{\approx_1,\approx_3}(F \circ_{\mathrm{SB}} G).$$

Let us conclude by giving some examples of tautologies describing a transfer of functionality property:

$$\mathrm{Tot}(\approx_1), \mathrm{Tot}(\approx_2) \models_{\mathrm{L}\forall^*} (F \subseteq_{\mathrm{SB}} G)^2 \to_* (\mathrm{Func}_{\approx_1,\approx_2}(F) \to_{\mathrm{B}} \mathrm{Func}_{\approx_1,\approx_2}(G \cap_{\mathrm{B}} \mathrm{dom}\, F)),$$
$$\mathrm{Tot}(\approx_1), \mathrm{Tot}(\approx_2) \models_{\mathrm{L}\forall^*} (F \subseteq_{\mathrm{SS}} G)^2 \to_* (\mathrm{Func}_{\approx_1,\approx_2}(F) \to_{\mathrm{S}} \mathrm{Func}_{\approx_1,\approx_2}(G)),$$

where $\varphi^2 \equiv_{\mathrm{df}} \varphi \mathbin{\&_{\mathrm{B}}} \varphi$ and $(\mathrm{dom}\, F)(x,y) \equiv_{\mathrm{df}} {!F(x,y)}$. Replacing $\to_*$ by $\to^*$ in the second formula leads again to tautology, but it is not so for the first one.

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Omitting Types Theorem in Mathematical Fuzzy Logic*

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Fuzzy structures constitute the semantics of first-order Mathematical Fuzzy Logic. These structures can be seen as generalizations of the classical ones by interpreting predicate symbols as functions from the domain to arbitrary lattice-ordered sets, instead of the two element Boolean algebra. Model theory of fuzzy structures is an evolving branch of Mathematical Fuzzy Logic which investigates, among others, how results from classical model theory are transformed within this fuzzy framework [6]. In this talk we focus on Omitting Types Theorem and investigate its analog in the framework of Mathematical Fuzzy logic.

In the classical model theory, a *type* is a syntactic object – a set of formulas – which shows how the elements of structures might behave. For a type and a structure, a basic question is whether there is an element of the structure 'described' by the type; if yes, the type is called *realized*, otherwise *omitted*. For example, let T be the theory of Peano arithmetic and let Σ be the set of formulas $\{0 \neq x, S0 \neq x, SS0 \neq x, \ldots\}$. The standard model of T omits Σ, while all the nonstandard models of T realize Σ (see, e.g., [4]). For a type Σ and a theory T, the Omitting Types Theorem gives a sufficient condition for the existence of a model of T omitting Σ.

Omitting types theorem can be investigated in different settings, e.g., for continuous logic or in the context of model theory of metric structures [3,9]. Previous attempts [2,16] to formulate and prove an analog of this theorem in the realm of Mathematical Fuzzy Logic were restricted to (variants) of Łukasiewicz logic [12,15,17]. To investigate the Omitting Types Theorem for wider classes of fuzzy logics, one has to look into its key ingredient – the notion of *isolation*.

In classical logic, a type Σ (i.e., a set of formulas with free variables) is *isolated* in a theory T (i.e., a set of sentences) if there exists a formula $\delta(\bar{x})$ such that

a) $T \cup \{\delta(\bar{x})\}$ is satisfiable, and

b) $T \models \delta(\bar{x}) \to \sigma(\bar{x})$, for all $\sigma(\bar{x}) \in \Sigma$.

This simple and elegant condition is really 'tailored' for the use in classical logic, and a more complex notion is needed for the fuzzy setting.

In order to formulate our notion of isolation we need to resort to a notion of consequence common in classical model theory but (up to recently) mostly ignored in the fuzzy logic literature: for a set of formulas Γ, we say that a formula φ is a *consequence* of Γ, written $\Gamma \models^l \varphi$, if for each safe structure $\mathfrak{M}$ and for each $\mathfrak{M}$-evaluation v, if all formulas of Γ are valid[1] in $\mathfrak{M}$ under the evaluation v, then so is φ.

Definition 1 (Isolation). *A type Σ is* isolated *in a theory T if there are formulas $\varphi(\bar{x}, \bar{y})$ and $\tau(\bar{x}, \bar{y})$ such that*

1. $T, \varphi(\bar{x}, \bar{y}) \not\models^l \tau(\bar{x}, \bar{y})$,

2. $T, \varphi(\bar{x}, \bar{y}) \models^l \sigma(\bar{x}) \vee \tau(\bar{x}, \bar{y})$, for all $\sigma(\bar{x}) \in \Sigma$.

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[1] The definition of *validity* depends on the fuzzy logic in question; the most common ones are being equal to the top of the lattice of truth values or being equal to or greater than a selected least designated element.

For classical logic both notions of isolation coincide. Indeed, if a type satisfies conditions a) and b) then taking τ to be $\overline{0}$, we see that it is isolated in the sense of Definition 1. Conversely, if we are in classical logic and a type is isolated in the sense of Definition 1, i.e., there are formulas $\varphi(\bar{x}, \bar{y})$ and $\tau(\bar{x}, \bar{y})$ such that conditions 1) and 2) are satisfied, then the type is isolated also in the classical sense by taking $\delta(\bar{x}) = (\exists \bar{y})(\varphi(\bar{x}, \bar{y}) \wedge \neg\tau(\bar{x}, \bar{y}))$.

Apart from the notion of isolation, there is another problem caused by our non-classical setting: in classical logic one could construct the canonical model of a Henkin theory whose elements 'are' just the constants newly added in the construction due to the fact that for any closed term t we have $(\exists x)(x = t)$ and so there is a 'new' constant $c = t$. As in a language without equality we cannot do the same trick, during the construction we have to make sure that the type is not 'realized' by some already existing closed term. Therefore we must assure that there are not too many closed terms occurring in axioms of T and so our variant of the Omitting types theorem is rather complex:

Theorem 1 (Omitting types [5])**.** *Let* L *be an axiomatic expansion of the Uninorm Logic UL such that for all formulas* φ, ψ, χ*:*

$$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} \chi \to \chi' \qquad \text{where } \chi' \text{ results from } \chi \text{ by replacing its subformula } \varphi \text{ by } \psi.$$

Furthermore let $\mathcal{P}$ *be a countable predicate language with only nullary function symbols,* T *a consistent theory such that at most finitely many of its elements involve object constants, and* Σ *a non-isolated type over* T*. Then there is a countable model of* T *which omits* Σ*.*

The class of logics covered by this theorem contains the most prominent fuzzy logics such as the Esteva and Godo's logic MTL of left-continuous t-norms [10], Hájek's logic BL of continuous t-norms, Łukasiewicz Logic, Gödel logic, Metcalfe and Montagna's Uninorm Logic UL [13], classical logic, all core fuzzy logics of Hájek and Cintula [11].

Let us conclude by a few remarks and comments of possible future research in this area:

- One could easily observe that we could generalize Theorem 1 in such a way that the required model of T would omit countably many non-isolated types at once.
- Working in logics with crisp equality would allow us to drop the finiteness restriction in Theorem 1 and prove it for predicate languages involving general function symbols; however this kind of fuzzy logics are relatively unexplored.
- Our rather complex notion of isolation can be greatly simplified in classical setting; a natural question is if the same can be done (at least partly) in some strong fuzzy logics.
- In classical logic the isolated types over a complete theory are realized in all its models. This is clearly not the case in our setting; again, an interesting question would be to explore in which fuzzy logics (and for which notion of complete theory) the claim holds.
- We have identified several properties a logic has to satisfy for our proof of Theorem 1 to go through. All but one of them can be proved for much wider classes of fuzzy logics (e.g., for core semilinear logics of [7] or for finitary weakly implicative lattice-disjunctive semilinear logics of [8]). The problematic property is called *preSkolemization*: for any theory T, formulas $\varphi(x), \psi$, and constant symbol c not occurring in $T \cup \{\varphi(x), \psi\}$,

  $$T, (\exists x)\varphi(x) \vdash \psi \quad \text{iff} \quad T, \varphi(c) \vdash \psi.$$

  Interestingly enough, this property holds in many of the logics mentioned earlier at least in a restricted setting (e.g., in logics with Baaz–Monteiro $\triangle$ connective [1,14], it holds for all formulas starting with $\triangle$). It would be interesting to see if it would allow us to prove our result either for all types, or at least for some special ones.

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Skolemization and Herbrand Theorems for Lattice-Valued Logics*

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In classical first-order logic, questions of validity and semantic consequence reduce to the satisfiability of a set of sentences; Skolemization methods and Herbrand theorems then further reduce these questions to the satisfiability of a set of propositional formulas (see, e.g., [4]). For first-order non-classical logics, the situation is not so straightforward. First, due to the absence of certain quantifier shifts, formulas are not always equivalent to prenex formulas, and, second, semantic consequence does not (typically) reduce to satisfiability. Hence (non-prenex) sentences should be considered separately as premises and conclusions of consequences. General Skolemization methods and Herbrand theorems therefore take various forms, applying either to the left or right of the consequence relation, and to restricted sets of formulas.

Skolemization procedures can be more carefully defined to replace strong occurrences of quantifiers in subformulas on the left, and weak occurrences on the right; however, satisfiability or, more generally, semantic consequence, may not be preserved. For example, in first-order intuitionistic logic, formulas such as $\neg\neg(\forall x)P(x) \to (\forall x)\neg\neg P(x)$ do not Skolemize (see, e.g., [1], also for methods for addressing these problems). An alternative solution is provided by the more general "parallel Skolemization" procedure developed in [2, 5]. The key idea is to remove strong occurrences of quantifiers on the left of the consequence relation and weak occurrences of quantifiers on the right by introducing disjunctions and conjunctions, respectively, of formulas with multiple new function symbols. In particular, a sentence $(\forall\bar{x})(\exists y)\varphi(\bar{x}, y)$ occurring as the conclusion of a consequence is rewritten for some $n \in \mathbb{N}^+$ as $(\forall\bar{x})\bigvee_{i=1}^{n}\varphi(\bar{x}, f_i(\bar{x}))$ where each function symbol f_i is new for $i = 1 \ldots n$. This method has been used by Baaz and Iemhoff in [2] to establish Skolemization results for first-order intermediate logics whose Kripke models (with or without the constant domains condition) admit a finite model property, and by the current authors in [5] to obtain similar results for a wide range of substructural and many-valued logics.

In the work reported here, we introduce a general framework for the study of first-order non-classical logics based on algebras with a (complete) lattice reduct and additional connectives that are monotone or antitone in each argument. Such logics include first-order fuzzy logics, intermediate logics, exponential-free linear logic, relevance logics, and logics without contraction (see, e.g., [7–9, 11, 12]). To obtain the desired generality, we consider consequence relations on inequalities between formulas, obtaining an obvious reduction to formulas in the presence of suitable connectives.

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We study Herbrand and Skolemization theorems in this setting and obtain general results covering and extending existing results from [1, 3, 5, 6]. In particular, we prove a Herbrand theorem for logics satisfying a finitarity condition for certain "propositional consequences" and show that under certain additional assumptions, this finitarity condition is necessary for the theorem to hold. We also establish a parallel Skolemization property for logics admitting variants of the witnessed model property introduced by Hájek in [10] and, that for finitary logics, this witnessed model property is equivalent to parallel Skolemization. We also consider logics that satisfy only a weaker witnessing property and admit parallel Skolemization for prenex sentences.

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Residuated structures with functional frames

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A *residuation algebra* is an expansion of a bounded distributive lattice A by a pair of binary division operations $\backslash$ and $/$ such that

- $\backslash$ and $/$ preserve finite meets occurring in their numerators, and
- for all $a, b, c \in A$,
$$b \leq a\backslash c \iff a \leq c/b$$

Residuation algebras abound in algebraic logic, where the operations $\backslash$ and $/$ interpret implication connectives arising as the residuals of groupoid operations in bounded, distributive residuated lattice-ordered groupoids. Such algebraic structures (and the logics for which they provide semantics) often admit fruitful duality-theoretic investigation, in which the operations $\backslash$ and $/$ are encoded on duals by a ternary relation R (see, e.g., [5]). In the best cases, R may be taken as the graph of a (possibly partial) function. For example, residuation algebras that satisfy the distribution law
$$a\backslash(b \vee c) = (a\backslash b) \vee (a\backslash c),$$
have functional duals in the aforementioned sense. Semilinear commutative residuated lattices (i.e., subdirect products of totally-ordered commutative residuated lattices) satisfy this distribution law, and consequently residuated structures with functional duals include: Gödel algebras, Sugihara monoids, MTL-algebras, and in particular MV-algebras and BL-algebras. Lattice-ordered groups and lattice-ordered loops are examples of residuated structures satisfying the above distribution law and yet failing to be semilinear in general. Due to the relative simplicity of binary operations as opposed to ternary relations, functionality affords an avenue of duality-theoretic inquiry that the complexity of ternary relational structures may obstruct in other situations (see, e.g., [2]).

Residuation algebras with functional duals were studied in the context of automata theory in [3], where Gehrke employs extended Stone-Priestley duality [5] in order to capture topological algebras[1] as dual spaces. Among other things, Gehrke characterizes (see [3, Proposition 3.16]) those residuation algebras $(A, \backslash, /)$ for which the relation corresponding to $\backslash$ and $/$ under the Stone-Priestley duality is functional. Gehrke's investigation of functionality is conducted in the context of topological duality for distributive lattices, and makes no explicit use of the theory of canonical extensions. Moreover, her characterization is second-order, and she leaves open the question of whether there is any first-order condition in the language of residuation algebras that characterizes functionality on duals.

Motivated by this question, the present study treats the functionality of the duals of residuation algebras in the context of their canonical extensions (see, e.g., [4]). Given a residuation

[1]Given an algebraic similarity type τ, a topological algebra of type τ is an algebra of type τ in the category of topological spaces, i.e. it is a topological space enriched with continuous operations for each $f \in \tau$.

algebra $\mathbf{A} = (A, \backslash, /)$, as is standard in the theory of canonical extensions we define its *relational dual structure* $\mathbf{A}_+^\delta = (J^\infty(A^\delta), \geq, R)$, where $J^\infty(A^\delta)$ is the collection of completely join-irreducible elements of the canonical extension A^δ of A, and R is a ternary relation on $J^\infty(A^\delta)$ defined by

$$R(x, y, z) \iff x \leq y \cdot z,$$

where $\cdot$ is the common residual of the π-extensions $\backslash^\pi$ and $/^\pi$ of $\backslash$ and $/$ in A^δ. We say that R is *functional* provided that $y \cdot z \in J^\infty(A^\delta) \cup \{\bot\}$ whenever $y, z \in J^\infty(A^\delta)$. In this event, we also say that $\mathbf{A}_+^\delta$ is functional. Our approach yields the following.

First, moving to the environment of canonical extensions provides a more transparent and modular account of how the validity of the distribution law $a\backslash(b \vee c) = (a\backslash b) \vee (a\backslash c)$ guarantees the functionality of the dual relation. Using the fact that this identity is canonical (see [1]), we show that the validity of this distribution law forces the product of join-irreducible elements (where product is given by the common residual of $\backslash^\pi$ and $/^\pi$ in the canonical extenion) to be either $\bot$ or finitely join-prime. Prime closed elements of the canonical extension are in turn completely join-irreducible. We obtain the functionality of the dual relation as a consequence of these two facts, only the first of which depends on the validity of the aforementioned distributive law.

Second, we offer a partial answer to the motivating question. Although our analysis of the distribution law $a\backslash(b \vee c) = (a\backslash b) \vee (a\backslash c)$ entails that there are large varieties of residuation algebras whose members have functional duals, we establish that there is no equational or quasiequational condition that characterizes the functionality of duals as a consequence of the following.

Theorem 1. *There is no collection of universal first-order sentences Σ in the language of residuation algebras such that for each residuation algebra $\mathbf{A}$, $\mathbf{A} \models \Sigma$ if and only if $\mathbf{A}_+^\delta$ is functional.*

Third, our use of the theory of canonical extensions allows us to present an entirely algebraic characterization of functionality on duals in the spirit of [3, Proposition 3.16], freeing this characterization of particular duality-theoretic representations. Specifically, we obtain the following result.

Theorem 2. *The following conditions are equivalent for any residuation algebra $\mathbf{A} = (A, \backslash, /)$:*

1. *The relational structure $\mathbf{A}_+^\delta$ is functional.*
2. $\forall a, b, c \in A, \forall x \in J^\infty(A^\delta)[x \leq a \Rightarrow \exists a'[a' \in L \mathbin{\&} x \leq a' \mathbin{\&} a\backslash(b \vee c) \leq (a'\backslash b) \vee (a'\backslash c)]$.
3. *For all $x \in J^\infty(A^\delta)$, the map $x\backslash^\pi(\_) : O(A^\delta) \to O(A^\delta)$ is $\vee$-preserving, where $O(A^\delta)$ denotes the join-closure of A in A^δ.*

As an added benefit, our approach to functionality via canonical extensions sheds light on the phenomenon of *totality*, i.e., where the dual relation corresponding to the division operations is not only a functional relation, but also a total function.

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$\mathbf{IUL}^{fp}$ Enjoys Finite Strong Standard Completeness *

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Abstract

Hahn's celebrated embedding theorem asserts that linearly ordered Abelian groups embed in the lexicographic product of real groups. In this talk the partial-lexicographic product construction is introduced, a class of residuated monoids, namely, group-like FL_e-chains which possess only a finite number of idempotents are represented as partial-lexicographic products of linearly ordered Abelian groups, and as a corollary, Hahn's theorem is extended to this residuated monoid class by showing that any such algebra embeds in some partial-lexicographic product of linearly ordered Abelian groups. By relying on this embedding theorem we can prove that the logic $\mathbf{IUL}^{fp}$ enjoys finite strong standard completeness.

Mathematical fuzzy logics have been introduced in [5], and the topic is a rapidly growing field ever since. Substructural fuzzy logics were introduced in [12] as substructural logics that are standard complete, that is, complete with respect to algebras whose lattice reduct is the real unit interval $[0, 1]$. Also, standard completeness for several substructural logics, all are stronger than the there-introduced Uninorm Logic ($\mathbf{UL}$), has been proven, with the notable exception of the Involutive Uninorm Logic ($\mathbf{IUL}$). Its standard completeness has remained an open problem, which has withstood the attempts using the usual embedding method of [7] or the density elimination technique of [12]. An algebraic semantics for $\mathbf{IUL}$ is the variety of bounded involutive FL_e-chains. The class of bounded involutive FL_e-chains is quite rich; even its integral algebras over $[0, 1]$, the standard IMTL-chains is a class containing, e.g., the connected rotations [6] of all standard MTL-chains.

Group-like FL_e-chains are involutive FL_e-chains satisfying the condition that the unit of the monoidal operation coincides with the constant that defines the order-reversing involution $'$; in notation $t = f$. Since for any involutive FL_e-chain $t' = f$ holds, one extremal situation is the integral case, that is, when t is the top element of the universe and hence f is its bottom one, and the other extremal situation is the group-like case when the two constants coincide. Group-like FL_e-chains constitute the algebraic semantics for Involutive Uninorm Logic with Fixed Point ($\mathbf{IUL}^{fp}$), which was introduced in [11]. Prominent examples of group-like FL_e-algebras are lattice-ordered Abelian groups and odd Sugihara algebras, the latter constitute an algebraic semantics of $\mathbf{IUML}^*$, which is a logic at the intersection of relevance logic and fuzzy logic [3]. These two examples are extremal in the sense that lattice-ordered Abelian groups have a single idempotent element, namely the unit element, whereas all elements of any odd Sugihara algebra are idempotent.

In order to make a step in the direction of settling the standard completeness problem for $\mathbf{IUL}$ in the algebraic manner of [7], and in order to narrow the gap between the two extremal classes mentioned above, in this talk we gain a deeper knowledge about the class of involutive FL_e-chains by focusing first on group-like FL_e-chains. For those group-like FL_e-chains, which have finitely many idempotents, a representation theorem will be proven, using only linearly ordered Abelian groups as building blocks and the here-defined partial-lexicographic product construction. The naturally ordered condition or its dual notion, called divisibility, has always been a postulate in previous structural descriptions of classes of residuated lattices, see e.g. [10, 13]. An exception is in [8, 9], where only a weakened form of the naturally ordered condition, called absorbent continuity, has been assumed. On the other hand, as it

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was pointed out in [3], non-integral residuated structures, and consequently, substructural logics without the weakening rule, are far less understood at present then their integral counterparts. Therefore, from a general viewpoint, the algebraic part of our study is a contribution to non-integral residuated structures without postulating even a weakened form of the divisibility condition. Hahn's embedding theorem states that linearly ordered Abelian groups embed in some lexicographic product of real groups [4]. Conrad, Harvey, and Holland generalized Hahn's theorem for lattice-ordered Abelian groups in [1]. As a corollary of our representation theorem, we extend Hahn's theorem to those group-like FL_e-chains which possess finitely many idempotents. The price for not having inverses will be that the embedding is made into partial-lexicographic products rather than lexicographic ones; the building blocks (linearly ordered Abelian groups) remain the same. We also claim that every finitely generated group-like FL_e-chain possesses only finitely many idempotents. By relying on this and our embedding theorem we can prove that the logic $\mathbf{IUL}^{fp}$ enjoys finite strong standard completeness.

Let $(X, \leq)$ be a chain (a linearly ordered set). For $x \in X$ define the predecessor $x_\downarrow$ of x to be the maximal element of the set of elements which are smaller than x, if it exists, define $x_\downarrow = x$ otherwise. Define $x_\uparrow$ dually. An *FL_e-algebra*[1] is a structure $(X, \wedge, \vee, \circledast, \rightarrow_\circledast, t, f)$ such that $(X, \wedge, \vee)$ is a lattice, $(X, \leq, \circledast, t)$ is a commutative, residuated[2] monoid, and f is an arbitrary constant. One defines $x' = x \rightarrow_\circledast f$ and calls an FL_e-algebra *involutive* if $(x')' = x$ holds. We call an FL_e-algebra *group-like* if it is involutive and $t = f$.

Definition 1. (Partial Lexicographic Products) Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \rightarrow_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \circledast, \rightarrow_\circledast, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement ${}^{\overset{*}{\prime}}$ and ${}'$, respectively.

1. Add a new element $\top$ to Y as a top element, and extend $\circledast$ by $\top \circledast y = y \circledast \top = \top$ for $y \in Y \cup \{\top\}$, then add a new element $\perp$ to $Y \cup \{\top\}$ as a bottom element, and extend ${}'$ by $\perp' = \top$, $\top' = \perp$ and $\circledast$ by $\perp \circledast y = y \circledast \perp = \perp$ for $y \in Y \cup \{\top, \perp\}$. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be cancellative subalgebras of $\mathbf{X}$ such that $\mathbf{X}_2 \leq \mathbf{X}_1$. We define $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, {}^{\top\perp})}^{\mathbf{\Gamma}(\mathbf{X_2}, \mathbf{Y})} = \left(X_{\Gamma(X_1, {}^{\top\perp})}^{\Gamma(X_2, Y)}, \leq, \diamond, \rightarrow_\diamond, (t_X, t_Y), (f_X, f_Y) \right)$, where

$$X_{\Gamma(X_1, {}^{\top\perp})}^{\Gamma(X_2, Y)} = (X_1 \times \{\top, \perp\}) \cup (X_2 \times Y) \cup ((X \setminus X_1) \times \{\perp\}),$$

$\leq$ is the restriction of the lexicographical order of $\leq_X$ and $\leq_{Y \cup \{\top, \perp\}}$ to $X_{\Gamma(X_1, {}^{\top\perp})}^{\Gamma(X_2, Y)}$, $\diamond$ is defined coordinatewise, and the operations ${}^{\overset{\diamond}{\prime}}$ and $\rightarrow_\diamond$ are given by

$$(x_1, y_1) \rightarrow_\diamond (x_2, y_2) = \left((x_1, y_1) \diamond (x_2, y_2)^{\overset{\diamond}{\prime}} \right)^{\overset{\diamond}{\prime}} \text{ and } (x, y)^{\overset{\diamond}{\prime}} = \begin{cases} (x^{\overset{*}{\prime}}, y') & \text{if } x \in X_1 \\ (x^{\overset{*}{\prime}}, \perp) & \text{if } x \notin X_1 \end{cases}.$$

Call $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, {}^{\top\perp})}^{\mathbf{\Gamma}(\mathbf{X_2}, \mathbf{Y})}$ the *(type III) partial-lexicographic product* of X, X_1, X_2, and Y.

In particular, if $\mathbf{X}_1 = \mathbf{X}_2$ then call $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, {}^{\top\perp})}^{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y})}$ the *(type I) partial-lexicographic product* of X, X_1, and Y, and denote it by $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\top\perp})}$.

2. Add a new element $\top$ to Y as a top element, and extend $\circledast$ by $\top \circledast y = y \circledast \top = \top$ for $y \in Y \cup \{\top\}$. Let $\mathbf{X}_1$ be a linearly ordered, discretely embedded[3], prime[4] and cancellative subalgebra of $\mathbf{X}$[5], and

[1] Other terminologies for FL_e-algebras are: pointed commutative residuated lattices or pointed commutative residuated lattice-ordered monoids.

[2] That is, there exists a binary operation $\rightarrow_\circledast$ such that $x \circledast y \leq z$ if and only if $x \rightarrow_\circledast z \geq y$; this equivalence is called residuation condition or adjointness condition, $(\circledast, \rightarrow_\circledast)$ is called an adjoint pair. Equivalently, for any x, z, the set $\{v \mid x \circledast v \leq z\}$ has its greatest element, and $x \rightarrow_\circledast z$ is defined as this element: $x \rightarrow_\circledast z := \max\{v \mid x \circledast v \leq z\}$.

[3] We mean that for $x \in X_1$, it holds true that $x \notin \{x_\uparrow, x_\downarrow\} \subset X_1$ ($\downarrow$ and $\uparrow$ are computed in X).

[4] We mean that $(X \setminus X_1) * (X \setminus X_1) \subseteq X \setminus X_1$ holds.

[5] Equivalently, let $\mathbf{X}_1 = \mathbf{gr}(\mathbf{X})$, which is the subalgebra of $\mathbf{X}$ over $\{x \in X \mid x \text{ has inverse}\}$ and called the group part of $\mathbf{X}$, provided that $\mathbf{gr}(\mathbf{X})$ is linearly ordered and discretely embedded into $\mathbf{X}$.

let $\mathbf{X}_2 \leq \mathbf{X}_1$. We define $\mathbf{X}_{\mathbf{\Gamma(X_1,\top)}}^{\mathbf{\Gamma(X_2,Y)}} = \left(X_{\Gamma(X_1,\top)}^{\Gamma(X_2,Y)}, \leq, \diamond, \rightarrow_\diamond, (t_X, t_Y), (f_X, f_Y)\right)$, where

$$X_{\Gamma(X_1,\top)}^{\Gamma(X_2,Y)} = (X_1 \times \{\top\}) \cup (X_2 \times Y) \cup ((X \setminus X_1) \times \{\top\})$$

$\leq$ is the restriction of the lexicographical order of $\leq_X$ and $\leq_{Y\cup\{\top\}}$ to $X_{\Gamma(X_1,\top)}^{\Gamma(X_2,Y)}$, $\diamond$ is defined coordinatewise, and the operations $^{\prime\diamond}$ and $\rightarrow_\diamond$ are given by

$$(x_1, y_1)\rightarrow_\diamond(x_2, y_2) = \left((x_1, y_1) \diamond (x_2, y_2)^{\prime\diamond}\right)^{\prime\diamond} \text{ and } (x, y)' = \begin{cases} (x^{\prime*}, \top) & \text{if } x \notin X_1 \text{ and } y = \top \\ (x^{\prime*}, y') & \text{if } x \in X_1 \text{ and } y \in Y \\ ((x^{\prime*})_\downarrow, \top) & \text{if } x \in X_1 \text{ and } y = \top \end{cases}.$$

Call $\mathbf{X}_{\mathbf{\Gamma(X_1,\top)}}^{\mathbf{\Gamma(X_2,Y)}}$ the *(type IV) partial-lexicographic product* of X, X_1, X_2, and Y.
In particular, if $\mathbf{X}_1 = \mathbf{X}_2$ then call $\mathbf{X}_{\mathbf{\Gamma(X_1,\top)}}^{\mathbf{\Gamma(X_1,Y)}}$ the *(type II) partial-lexicographic product* of X, X_1, and Y, and denote it by $\mathbf{X}_{\mathbf{\Gamma(X_1,Y^\top)}}$.

Lemma 1. *Adopt the notation of Definition 1. Then $\mathbf{X}_{\mathbf{\Gamma(X_1,\top\bot)}}^{\mathbf{\Gamma(X_2,Y)}}$ and $\mathbf{X}_{\mathbf{\Gamma(X_1,\top)}}^{\mathbf{\Gamma(X_2,Y)}}$ are involutive FL$_e$-algebras with the same rank as that of $\mathbf{Y}$.*[6] *$\mathbf{X}_{\mathbf{\Gamma(X_1,\top\bot)}}^{\mathbf{\Gamma(X_2,Y)}} \leq \mathbf{X}_{\mathbf{\Gamma(X_1,Y^{\top\bot})}}$ and $\mathbf{X}_{\mathbf{\Gamma(X_1,\top)}}^{\mathbf{\Gamma(X_2,Y)}} \leq \mathbf{X}_{\mathbf{\Gamma(X_1,Y^\top)}}$ hold. In particular, if $\mathbf{Y}$ is group-like then so are $\mathbf{X}_{\mathbf{\Gamma(X_1,\top\bot)}}^{\mathbf{\Gamma(X_2,Y)}}$ and $\mathbf{X}_{\mathbf{\Gamma(X_1,\top)}}^{\mathbf{\Gamma(X_2,Y)}}$.*

Theorem 1. *For a group-like FL$_e$-algebra $(X, \wedge, \vee, \circledast, \rightarrow_\circledast, t, f)$, $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered abelian group if and only if $\circledast$ is cancellative.*

Theorem 2. (Group Representation) *If $\mathbf{X}$ is a group-like FL$_e$-chain, which has only $n \in \mathbf{N}$, $n \geq 1$ idempotents in its positive cone then there exist linearly ordered abelian groups $\mathbf{G}_i$ ($i \in \{1, 2, \dots, n\}$), $\mathbf{H}_{1,2} \leq \mathbf{H}_{1,1} \leq \mathbf{G}_1$, $\mathbf{H}_{i,2} \leq \mathbf{H}_{i,1} \leq \mathbf{\Gamma}(\mathbf{H}_{i-1,2}, \mathbf{G}_i)$ ($i \in \{2, \dots, n-1\}$), and a binary sequence $\iota \in \{\top\bot, \top\}^{\{2,\dots,n\}}$ such that $\mathbf{X} \simeq \mathbf{X}_n$, where $\mathbf{X}_1 := \mathbf{G}_1$ and $\mathbf{X}_i := \mathbf{X}_{i-1}{}_{\mathbf{\Gamma(H_{i-1,1},\iota_i)}}^{\mathbf{\Gamma(H_{i-1,2},G_i)}}$ ($i \in \{2, \dots, n\}$).*
[7]

Theorem 3. *The logic $\mathbf{IUL}^{fp}$ enjoys finite strong standard completeness.*

Sketch of the proof. Since IUL$^{\text{fp}}$-chains (that is, non-trivial bounded group-like FL$_e$-chains) constitute an algebraic semantics of $\mathbf{IUL}^{fp}$, we shall prove that any finitely generated IUL$^{\text{fp}}$-chain embeds into an IUL$^{\text{fp}}$-chain over $[0, 1]$. Thus we prove that any $\mathbf{IUL}^{fp}$ formula which is falsified in a linearly ordered model of finitely many $\mathbf{IUL}^{fp}$ formulas (which is always a finitely generated IUL$^{\text{fp}}$-chain) can also be falsified in a standard group-like FL$_e$-algebra, that is, in one over $[0, 1]$. To this end, let $\mathbf{Y}_1$ be a non-trivial finitely generated bounded group-like FL$_e$-chain, and let $\mathbf{Y}$ be the subalgebra of $\mathbf{Y}_1$ over its universe deprived its top and bottom elements. Then $\mathbf{Y}$ is a finitely generated (not necessarily bounded) group-like FL$_e$-chain. We can prove that any finitely generated group-like FL$_e$-chain has only a finite number of idempotents, thus $\mathbf{Y}$ has finitely many ($n \geq 1$) idempotents in its positive cone. Hence it has a type III-IV group representation by Theorem 2. Our plan is to embed $\mathbf{Y}$, guided by this group representation, into a group-like FL$_e$-chain $\mathbf{X}_n^*$ over a universe which is order isomorphic to $\mathbb{R}$. Then, using the order isomorphism together with an order isomorphism between $\mathbb{R}$ and $]0, 1[$, we can carry over the structure of $\mathbf{X}_n^*$ into $]0, 1[$, and finally we can add a top and a bottom element (as in item 1 of Definition 1) to get a group-like FL$_e$-chain over $[0, 1]$, in which $\mathbf{Y}_1$ embeds.

[6] The rank of an involutive FL$_e$-algebra is positive if $t > f$, negative if $t < f$, and 0 if $t = f$.
[7] In the spirit of Theorem 1 we identify linearly ordered abelian groups by cancellative, group-like FL$_e$-chains here.

To this end, let a group representation of $\mathbf{Y}$ be given, that is, $\mathbf{G}_i$ $(i \in \{1,2,\ldots,n\})$ linearly ordered abelian groups, $\mathbf{H}_{1,2} \leq \mathbf{H}_{1,1} \leq \mathbf{G}_1$ and $\mathbf{H}_{i,2} \leq \mathbf{H}_{i,1} \leq \mathbf{\Gamma}(\mathbf{H}_{i-1,2}, \mathbf{G}_i)$ $(2 \leq i \leq n)$, and $\iota \in \{\top\bot, \top\}^{\{2,\ldots,n\}}$ such that $\mathbf{Y} \simeq \mathbf{Y}_n$, where $\mathbf{Y}_1 = \mathbf{G}_1$ and $\mathbf{Y}_i = \mathbf{Y}_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1,1},\iota_i)}^{\mathbf{\Gamma}(\mathbf{H}_{i-1,2},\mathbf{G}_i)}$ $(2 \leq i \leq n)$. It can be shown that for $i \in \{1,\ldots,n\}$, qua group-like FL_e-chains,

$$\mathbf{G}_i \simeq \bigoplus_{i=1}^{k_i} \mathbb{Z}$$

holds for some $k_i \in \mathbf{N}$, where for $k_i = 0$ we mean that $\mathbf{G}_i$ is the one-element group. We define two series of group-like FL_e-chains as follows: For $j \in \mathbb{N}$, $j \geq 1$, let $\mathbf{Z}_0 = \mathbf{Z}_1 := \mathbb{Z}$, $\mathbf{Z}_{j+1} := \mathbb{Z}_{\mathbf{\Gamma}(\mathbb{Z},\mathbf{Z}_j^{\top})}$, $\mathbf{R}_0 = \mathbf{R}_1 := \mathbb{R}$, $\mathbf{R}_{j+1} := \mathbb{R}_{\mathbf{\Gamma}(\mathbb{Z},\mathbf{R}_j^{\top\bot})}$. Finally, for $i \in \{1,\ldots,n\}$ let

$$\mathbf{X}_i^* = \begin{cases} \mathbf{R}_{k_1} & \text{if } i = 1 \text{ and either } \iota_2 = \top\bot \text{ or } i = n \\ \mathbf{Z}_{k_1} & \text{if } i = 1,\ \iota_2 = \top \\ \mathbf{X}^*_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1,1},\mathbf{R}_{k_i}{}^{\top\bot})} & \text{if } 2 \leq i \leq n,\ \iota_i = \top\bot, \text{ and either } \iota_{i+1} = \top\bot \text{ or } i = n \\ \mathbf{X}^*_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1,1},\mathbf{Z}_{k_i}{}^{\top\bot})} & \text{if } 2 \leq i \leq n-1,\ \iota_i = \top\bot,\ \iota_{i+1} = \top \\ \mathbf{X}^*_{i-1}{}_{\mathbf{\Gamma}(\mathbf{gr}(\mathbf{X}^*_{i-1}),\mathbf{Z}_{k_i}{}^{\top})} & \text{if } 2 \leq i \leq n-1,\ \iota_i = \top,\ \iota_{i+1} = \top \\ \mathbf{X}^*_{i-1}{}_{\mathbf{\Gamma}(\mathbf{gr}(\mathbf{X}^*_{i-1}),\mathbf{R}_{k_i}{}^{\top})} & \text{if } 2 \leq i \leq n,\ \iota_i = \top \text{ and either } \iota_{i+1} = \top\bot \text{ or } i = n \end{cases}.$$

It can be shown that $\mathbf{Y}$ embeds into $\mathbf{X}_n^*$, and that $\mathbf{X}_n^*$ is order isomorphic to $\mathbb{R}$. □

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Partially-ordered multi-type algebras, display calculi and the category of weakening relations

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We define partially-ordered multi-type algebras and use them as algebraic semantics for multi-type display calculi that have recently been developed for several logics, including dynamic epistemic logic [7], linear logic[10], lattice logic [11], bilattice logic [9] and semi-De Morgan logic [8].

Multi-type algebras, also called many-sorted or heterogeneous algebras [1] have been studied in the setting of universal algebra and they have applications, e.g., as abstract data types in computer science [2] and in algebraic logic [4].

A multi-type algebra is of the form $\mathbb{A} = ((A_\tau)_{\tau\in\mathcal{T}}, \mathcal{F})$ where each $f \in \mathcal{F}$ is a function $f : A_{\tau_1} \times \cdots \times A_{\tau_n} \to A_\tau$ for some $\tau_1, \ldots, \tau_n, \tau \in \mathcal{T}$. The set of types $\mathcal{T}$ and the sequences $\tau_1, \ldots, \tau_n, \tau$ for each operation $f \in \mathcal{F}$ determine the signature Σ of the algebra. A partially-ordered multi-type algebra (pom-algebra for short) replaces the carrier sets A_τ by partially-ordered sets $(A_\tau, \leq_\tau)$ and insists that the operations are order-preserving or order-reversing in each argument. This is recorded in the signature Σ by a sequence $\varepsilon \in \{1, \partial\}^n$ for each n-ary operation f such that $A^{\varepsilon_i}_{\tau_i} = A_{\tau_i}$ for $\varepsilon_i = 1$ if f is order-preserving in the ith coordinate, and $A^{\varepsilon_i}_{\tau_i} = A^{\partial}_{\tau_i}$, the dual poset, otherwise. Pom-algebras are a generalization of partially ordered (unitype) algebras. Varieties and quasivarieties of partially ordered algebras have been studied by Pigozzi in [12], and these universal algebraic concepts extend smoothly to pom-algebras of a given signature.

In the setting of this talk we mostly consider pom-algebras in which each carrier set has lattice operations $\vee_\tau, \wedge_\tau$ defined on it. In this case $\leq_\tau$ is assumed to be the lattice order and such algebras are called lattice-ordered multi-type algebras, or ℓm-algebras. An important insight of multi-type display calculi is that certain unitype lattice-ordered algebras can be recast as ℓm-algebras where each of the carriers support simpler algebraic structure. The decomposition of lattice-ordered unitype algebras into simpler loosely connected components can lead to the definition of uniform decision procedures, in the form of display calculi, for the equational theory or even the universal theory of the original unityped algebras. In some cases the unitype algebras satisfy identities that cannot be captured by display calculus rules, but for their pom-algebra counterparts this difficulty is resolved since the carriers of each type satisfy simpler identities.

Examples and results

Every lattice decomposes as a pom-algebra of a join-semilattice and a (disjoint) meet-semilattice, with the connection given by an inverse pair of order-isomorphisms. The decision procedure given by the display calculus of this variety of pom-algebras is Whitmann's solution of the word problem for free lattices, originally due to Skolem (see [3]).

A semi-De Morgan lattice (not necessarily distributive) decomposes as a lattice and a De Morgan lattice, connected by two unary order-preserving maps as in [8]. The display calculus for semi-De Morgan lattices interleaves Whitmann's solution for lattices with a similar algorithm for De Morgan lattices.

A linear logic algebra with exponentials decomposes into a residuated lattice and a Heyting algebra connected with two adjunctions. Again this leads to a display calculus for linear logic.

The concept of *residuated frame* from [6] is extended to ℓm-algebras and provides multi-type frame semantics for display calculi. This allows many of the techniques for residuated frames to be applied in the more general setting of ℓm-algebras.

Display calculi use sequents of the form $s \leq_\tau t$ as ingredients for the rules of the calculus, with both terms s, t having the same result type τ. However if $t = f(t_1, \ldots, t_n)$ one can also consider a new sequent symbol $s \leq_\tau^f (t_1, \ldots, t_n)$. In this case $\leq_\tau^f$ is a relation from A_τ to $A_{\tau_1}^{\varepsilon_1} \times \cdots \times A_{\tau_n}^{\varepsilon_n}$. Similarly the top-level operation symbol of the left-hand term of a sequent can be used to define a new sequent symbol. From this point of view sequent separator symbols are morphisms in the category of posets, similar to the adjunctions that map between carrier posets in pom-algebras.

For posets $\mathbb{P} = (P, \leq^{\mathbb{P}})$ and $\mathbb{Q} = (Q, \leq^{\mathbb{Q}})$ a binary relation $R \subseteq P \times Q$ is a *weakening relation* if $\leq^{\mathbb{P}} \circ R \circ \leq^{\mathbb{Q}} \subseteq R$. The class of posets with with weakening relations as morphisms forms a category $\overline{\mathsf{Pos}}$ that contains the category Pos of posets with order-preserving maps. We characterize the display calculus morphisms as weakening relations in the category $\overline{\mathsf{Pos}}$.

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A many-sorted polyadic modal logic

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Abstract

In [3], we defined a many-sorted polyadic modal logic together with its corresponding algebraic theory. The idea is not new: in [7, 8] two-sorted systems are analyzed and we used them as references for our approach, while in [1, 5] a general theory is developed by using a coalgebraic approach. However, to our knowledge, the general framework presented in this paper is new. In [2] a general many-sorted approach is developed, but the deductive system is different from ours. While the transition from the mono-sorted logic to a many-sorted one is a smooth process, we see our system as a step towards deepening the connection between modal logic and program verification, since our system can be seen as the propositional fragment of Matching logic, a first-order logic for specifying and reasoning about programs.

We present our system and its connections with logics used in program verification.

Our language is determined by a fixed, but arbitrary, many-sorted signature (S, Σ) and a set of many-sorted propositional variables $P = \{P_s\}_{s\in S}$ such that $P_s \neq \emptyset$ for any $s \in S$ and $P_{s_1} \cap P_{s_2} = \emptyset$ for any $s_1 \neq s_2$ in S. For any $n \in \mathbb{N}$ and $s, s_1, \ldots, s_n \in S$ we denote $\Sigma_{s_1\ldots s_n,s} = \{\sigma \in \Sigma \mid \sigma : s_1 \ldots s_n \to s\}$.

The transition from a mono-sorted to a many-sorted setting is a smooth process and we follow closely the developments from [4].

The set of formulas of $\mathcal{ML}_S$ is an S-indexed family $Form_S = \{Form_s\}_{s\in S}$ inductively defined as:

$$\phi_s ::= p \mid \neg\phi_s \mid {\phi_1}_s \vee {\phi_2}_s \mid \sigma({\phi_1}_{s_1}, \ldots, {\phi_n}_{s_n})$$

where $s \in S$, $p \in P_s$ and $\sigma \in \Sigma_{s_1\ldots s_n,s}$

We use the classical definitions of the derived logical connectors. For any $\sigma \in \Sigma_{s_1\ldots s_n,s}$ the *dual operation* is $\sigma^{\square}(\phi_1, \ldots, \phi_n) := \neg\sigma(\neg\phi_1, \ldots, \neg\phi_n)$.

In order to define the semantics we introduce the (S, Σ)-*frames* and the (S, Σ)-*models*. An (S, Σ)-*frame* is a tuple $\mathcal{F} = (\mathcal{W}, \mathcal{R})$ such that:

- $\mathcal{W} = \{W_s\}_{s\in S}$ is an S-sorted set of worlds and $W_s \neq \emptyset$ for any $s \in S$,
- $\mathcal{R} = \{\mathcal{R}_\sigma\}_{\sigma\in\Sigma}$ such that $\mathcal{R}_\sigma \subseteq W_s \times W_{s_1} \times \ldots \times W_{s_n}$ for any $\sigma \in \Sigma_{s_1\ldots s_n,s}$.

If $\mathcal{F}$ is an (S, Σ)-frame, then an (S, Σ)-*model based on* $\mathcal{F}$ is a pair $\mathcal{M} = (\mathcal{F}, \rho)$ where $\rho : P \to \mathcal{P}(\mathcal{W})$ is an S-*sorted valuation function* such that $\rho_s : P_s \to \mathcal{P}(W_s)$ for any $s \in S$. The model $\mathcal{M} = (\mathcal{F}, \rho)$ will be simply denoted as $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \rho)$. Following [4], if C is a set of frames then we say that a model $\mathcal{M}$ *is from* C if it is based on a frame from C.

Next we introduce a many-sorted *satisfaction relation*. If $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \rho)$ is an (S, Σ)-model, $s \in S$, $w \in W_s$ and $\phi \in Form_s$, then the many-sorted *satisfaction relation* $\mathcal{M}, w \models^{s} \phi$ is inductively defined as:

- $\mathcal{M}, w \models^{s} p$ iff $w \in \rho_s(p)$

- $\mathcal{M}, w \overset{s}{\models} \neg\psi$ iff $\mathcal{M}, w \overset{s}{\not\models} \psi$
- $\mathcal{M}, w \overset{s}{\models} \psi_1 \vee \psi_2$ iff $\mathcal{M}, w \overset{s}{\models} \psi_1$ or $\mathcal{M}, w \overset{s}{\models} \psi_2$
- if $\sigma \in \Sigma_{s_1 \ldots s_n, s}$, then $\mathcal{M}, w \overset{s}{\models} \sigma(\phi_1, \ldots, \phi_n)$ iff there exists $(w_1, \ldots, w_n) \in W_{s_1} \times \cdots \times W_{s_n}$ such that $\mathcal{R}_\sigma w w_1 \ldots w_n$ and $\mathcal{M}, w_i \overset{s_i}{\models} \phi_i$ for any $i \in [n]$.

Consequently, we define our logic. Let $\mathbf{K}_{(S,\Sigma)} = \{\mathbf{K}_s\}_{s \in S}$ be the least S-sorted set of formulas with the following properties:

(a0) for any $s \in S$, if $\alpha \in Form_s$ is a theorem in classical logic, then $\alpha \in \mathbf{K}_s$,

(a1) the following formulas are in $\mathbf{K}_s$

(K^i_σ) $\sigma^\square(\psi_1, \ldots, \phi \to \chi, \ldots, \psi_n) \to (\sigma^\square(\psi_1, \ldots, \phi, \ldots, \psi_n) \to \sigma^\square(\psi_1, \ldots, \chi, \ldots, \psi_n))$

$(Dual_\sigma)$ $\sigma(\psi_1, \ldots, \psi_n) \leftrightarrow \neg\sigma^\square(\neg\psi_1, \ldots, \neg\psi_n)$

for any $n \geq 1$, $i \in [n]$, $\sigma \in \Sigma_{s_1 \cdots s_n, s}$, $\psi_{s_1} \in Form_{s_1}, \ldots, \psi_{s_n} \in Form_{s_n}$ and $\phi, \chi \in Form_{s_i}$.

It is straightforward that $\mathbf{K}_{(S,\Sigma)}$ is a generalization of the modal system $\mathbf{K}$ (see [4] for the mono-sorted version). If $\Lambda \subseteq Form_S$ is an S-sorted set of formulas, then the *normal modal logic* defined by Λ is $\mathbf{K}\Lambda = \{\mathbf{K}\Lambda_s\}_{s \in S}$ where

$$\mathbf{K}\Lambda_s := \mathbf{K}_s \cup \{\lambda' \in Form_s \mid \lambda' \text{ is obtained by uniform substitution applied to a formula } \lambda \in \Lambda_s\}$$

Next, we assume $\Lambda \subseteq Form_s$ is an S-sorted set of formulas and we investigate the normal modal logic $\mathbf{K}\Lambda$. Note that, in our approach, a logic is defined by its axioms.

The deduction rules are *Modus Ponens* and

$$\textit{Universal Generalization}: \quad \frac{\phi}{\sigma^\square(\phi_1, .., \phi, ..\phi_n)}$$

The distinction between local and global deduction from the mono-sorted setting is deepened in our version: *locally*, the conclusion and the hypotheses have the same sort, while *globally*, the set of hypotheses is a many-sorted set.

Definition 1. Assume that $n \geq 1$, $s_1, \ldots, s_n \in S$ and $\phi_i \in Form_{s_i}$ for any $i \in [n]$. The sequence $\phi_1, \ldots, \phi_n$ is a $\mathbf{K}\Lambda$-*proof for* ϕ_n if, for any $i \in [n]$, φ_i is in $\mathbf{K}\Lambda_{s_i}$ or φ_i is inferred from $\varphi_1, \ldots, \varphi_{i-1}$ using *modus ponens* and *universal generalization.* If ϕ has a proof in $\mathbf{K}\Lambda$, then we say that ϕ is a *theorem* and we write $\vdash^s_{\mathbf{K}\Lambda} \phi$ where s is the sort of ϕ.

If $s \in S$, $\Phi_s \subseteq Form_s$ and $\phi \in Form_s$, then we say that ϕ is *locally provable from* Φ_s *in* $\mathbf{K}\Lambda$ and we write $\Phi_s \vdash^s_{\mathbf{K}\Lambda} \phi$, if there are $\phi_1, \ldots, \phi_n \in \Phi_s$ such that $\vdash^s_{\mathbf{K}\Lambda} (\phi_1 \wedge \ldots \wedge \phi_n) \to \phi$. If $\Gamma \subseteq Form$ is an S-sorted set of formulas, we say that ϕ is a *globally provable* from Γ, and we write $\Gamma \Vdash_{\mathbf{K}\Lambda} \phi$, if there exists a sequence $\phi_1, \ldots, \phi_n$ such that $\phi_n = \phi$ and, for any $i \in [n]$, $\phi_i \in Form_{s_i}$ is an axiom or $\phi_i \in \Gamma_{s_i}$ or it is inferred from $\phi_1, \ldots, \phi_{i-1}$ using *modus ponens* and *universal generalization.*

We further present the *local deduction.*

Theorem 2. *(Local deduction theorem for* $\mathbf{K}\Lambda$*) For any* $s \in S$ *and* $\Phi_s \cup \{\varphi, \psi\} \subseteq Form_s$*:*

$$\Phi_s \vdash^s_{\mathbf{K}\Lambda} \varphi \to \psi \textit{ iff } \Phi_s \cup \{\varphi\}_s \vdash^s_{\mathbf{K}\Lambda} \psi.$$

Following closely the approach from [4], in order to study the canonical models and to prove the completeness theorem, we need to study the consistent sets. For any $s \in S$, we say that the set $\Phi_s \subseteq Form_s$ is (*locally*) $\mathbf{K}\Lambda$-*inconsistent* if $\Phi_s \overset{s}{\vdash}_{\mathbf{K}\Lambda} \bot_s$ and it is (*locally*) $\mathbf{K}\Lambda$-*consistent*, otherwise. Using the (locally) maximal consistent sets, we define the *canonical model* $\mathcal{M}_{\mathbf{K}\Lambda} = (\mathcal{W}^{\mathbf{K}\Lambda}, \{\mathcal{R}_\sigma^{\mathbf{K}\Lambda}\}_{\sigma\in\Sigma}, V^{\mathbf{K}\Lambda})$ as follows:

(1) for any $s \in S$, $\mathcal{W}_s^{\mathbf{K}\Lambda} = \{\Phi \subseteq Form_s \mid \Phi \text{ is maximal } \mathbf{K}\Lambda\text{-consistent}\}$,

(2) for any $\sigma \in \Sigma_{s_1\ldots s_n,s}, w \in W_s^{\mathbf{K}\Lambda}, u_1 \in W_{s_1}^{\mathbf{K}\Lambda}, \ldots, u_n \in W_{s_n}^{\mathbf{K}\Lambda}$ we define

$$\mathcal{R}_\sigma^{\mathbf{K}\Lambda} w u_1 \ldots u_n \text{ iff } (\psi_1, \ldots, \psi_n) \in u_1 \times \cdots \times u_n \text{ implies } \sigma(\psi_1, \ldots, \psi_n) \in w$$

(3) $V^{\mathbf{K}\Lambda} = \{V_s^{\mathbf{K}\Lambda}\}_{s\in S}$ is the valuation defined by

$$V_s^{\mathbf{K}\Lambda}(p) = \{w \in W_s^{\mathbf{K}\Lambda} | p \in w\} \text{ for any } s \in S \text{ and } p \in P_s.$$

Our completeness results generalize the ones from [4].

Theorem 3. *For any many-sorted signature (S, Σ) the normal modal logic $\mathbf{K}_{(S,\Sigma)}$ is strongly complete with respect to the class of all (S, Σ)-frames. For any $\Lambda \subseteq Form$ the normal modal logic $\mathbf{K}\Lambda$ is complete with respect to the canonical model.*

We also investigate the *global deduction*: we relate the local and the global deduction, we prove a version of the deduction theorem in the global setting, we define the *globally consistent* sets and we prove that any globally consistent set has a model.

In order to develop an algebraic approach, we generalize the boolean algebras with operators as follows.

Definition 4. An (S, Σ)-*boolean algebra with operators* ((S, Σ)-*BAO*) is a structure

$$\mathfrak{A} = (\{\mathcal{A}_s\}_{s\in S}, \{f_\sigma\}_{\sigma\in\Sigma})$$

where $\mathcal{A}_s = (A_s, \vee_s, \neg_s, 0_s)$ is a boolean algebra for any sort $s \in S$ and, for any $\sigma \in \Sigma_{s_1\ldots s_n,s}$, $f_\sigma : A_{s_1} \times \ldots \times A_{s_n} \to A_s$ satisfies the following properties:

(N) $f_\sigma(a_1, \ldots, a_n) = 0_s$ whenever $a_i = 0_{s_i}$ for some $i \in [n]$,
(A) $f_\sigma(a_1, \ldots, a_i \vee_{s_i} a_i', \ldots, a_n) = f_\sigma(a_1, \ldots, a_i, \ldots, a_n) \vee_s f_\sigma(a_1, \ldots, a_i', \ldots, a_n)$
for any $i \in [n]$.

We mention that similar structures were defined in [1] , but in that case the operators are unary operations while, in our setting, they have arbitrary arities. We prove the analogue of the Jónsson-Tarski theorem as well as the algebraic completeness for our systems of many-sorted modal logic.

Theorem 5. *(Algebraic completeness for $\mathbf{K}\Lambda$) If $s \in S$ and $\phi \in Form_s$ then $\overset{s}{\vdash}_{\mathbf{K}\Lambda} \phi$ if and only if $e_s(\phi) = 1_s$ in A_s for any (S, Σ)-BAO $\mathfrak{A}$ that is a model of $\mathbf{K}\Lambda$ and any assignment e in $\mathfrak{A}$.*

Finally, we relate our system with *matching logic* [6], a many-sorted first-order logic for program specification and verification. The completeness theorem for matching logic is proved using a interpretation in the first-order logic with equality. The starting point of our investigation was the representation of the (mono-sorted) polyadic modal logic as a particular system of matching logic in [6, Section 8]. Our initial goals were: to understand the propositional part of matching logic, to give a self-contained proof of the completeness theorem, to identify the algebraic theory and to investigate the relation with modal logic. The present system is an initial step in this direction.

Let (S, Σ) be a many-sorted signature. A *matching logic* (S, Σ)*-model* is

$$M = (\{M_s\}_{s\in S}, \{\sigma_M\}_{\sigma\in\Sigma})$$

where $\sigma_M : M_{s_1} \times \ldots \times M_{s_n} \to \mathcal{P}(M_s)$ for any $\sigma \in \Sigma_{s_1\ldots s_n,s}$. For $\sigma \in \Sigma_{s_1\ldots s_n,s}$ we define $R_\sigma \subseteq M_{s_1} \times \cdots \times M_{s_n}$ by $R_\sigma w w_1 \ldots w_n$ iff $w \in \sigma_M(w_1, \ldots, w_n)$. Hence, $\mathcal{M} = (\{M_s\}_{s\in S}, \{R_\sigma\}_{\sigma\in\Sigma})$ is an (S, Σ)-frame so we consider $\mathfrak{M} = (\{\mathfrak{P}(M_s)\}_{s\in S}, \{m_\sigma\}_{\sigma\in\Sigma})$ the full complex algebra determined by $\mathcal{M}$.

With respect to the relation between matching logic and our general many-sorted polyadic modal logic, we can state the following:

- any matching logic model can be associated with an (S, Σ)-frame and vice versa,
- the algebraic theory of matching logic is the theory of (S, Σ)-boolean algebras with operators,
- the system $\mathbf{K}_{(S,\Sigma)}$ can be seen as the propositional counterpart of matching logic.

Even if Matching logic was the starting point of our research, one of the main issues was to connect our logic with already existing systems of many-sorted modal logic. In [8] the author defines a sound and complete two-sorted modal logic for projective planes. This system is a particular case of our general framework. We refer to [3] for more examples of many-sorted developments of modal logic that can be connected with our approach.

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Epimorphisms in varieties of square-increasing residuated structures*

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We provide some sufficient conditions for a variety of residuated structures to possess the epimorphism surjectivity property. Our main result concerns *square-increasing commutative residuated lattices (SRLs)*, possibly with *involution (SIRLs)*.

An SRL $\mathbf{A} = \langle A; \wedge, \vee, \cdot, \to, e\rangle$ comprises a lattice $\langle A; \wedge, \vee\rangle$, a commutative monoid $\langle A; \cdot, e\rangle$ that is square-increasing ($x \leq x \cdot x$), and a binary operation $\to$ satisfying the *law of residuation*

$$x \cdot y \leq z \text{ iff } y \leq x \to z.$$

We may enrich the language of SRLs with an *involution* $\neg$ that satisfies

$$x = \neg\neg x \quad \text{and} \quad x \to \neg y = y \to \neg x,$$

thus obtaining SIRLs.

Our interest in residuated structures stems from the fact that they algebraize substructural logics [2]. In the framework of [2], the variety of all SRLs algebraizes the positive full Lambek calculus with the structural rules of exchange and contraction (but not weakening). The logical counterpart of the variety of SIRLs is sometimes denoted as $\mathbf{LR^t}$ by relevance logicians (to indicate that the algebras are lattices, but need not be distributive). Equivalently, $\mathbf{LR^t}$ adds the contraction rule to classical linear logic (without exponentials and bounds).

One obtains more familiar algebras by imposing additional demands on S[I]RLs. For example, distributive SIRLs are called *De Morgan monoids*, and they algebraize the relevance logic $\mathbf{R^t}$. *Brouwerian algebras* are integral SRLs (meaning $x \leq e$), in which case $\cdot$ coincides with $\wedge$. They are the negation-less subreducts of Heyting algebras; the latter algebraize intuitionistic propositional logic. Integral SIRLs are Boolean algebras.

Let K be a variety of algebras. A homomorphism $f : \mathbf{A} \to \mathbf{B}$, with $\mathbf{A}, \mathbf{B} \in \mathsf{K}$, is called a K-*epimorphism* if, whenever $\mathbf{C} \in \mathsf{K}$ and g and h are homomorphisms from $\mathbf{B}$ to $\mathbf{C}$ such that $g \circ f = h \circ f$, then $g = h$. All surjective homomorphisms in K are K-epimorphisms. We say K has the *epimorphism surjectivity (ES) property* if the converse holds.

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When a variety K algebraizes a logic $\vdash$, then K has the ES property if and only if $\vdash$ has the *infinite (deductive) Beth (definability) property* [3], i.e., whenever a set Z of variables is defined *implicitly* in terms of a disjoint set X of variables by means of some set Γ of formal assertions about $X \cup Z$, then Γ also defines Z *explicitly* in terms of X. The same demand on $\vdash$, restricted to finite sets Z, is called the *finite Beth property*, and it amounts to the so-called *weak ES property* for K, i.e., every almost-onto K-epimorphism is surjective. (A homomorphism $h : \mathbf{A} \to \mathbf{B}$ is *almost-onto* if $\mathbf{B}$ is generated by $h[A] \cup \{b\}$ for some $b \in B$.)

Every variety of Brouwerian or Heyting algebras has the weak ES property [4]. In the presence of the weak ES property, the *amalgamation* property implies the ES property, as is well known. The amalgamation and ES properties are independent in the present context, however, and many varieties of (non-integral) S[I]RLs lacking the weak ES property have been identified (beginning with [8]).

Given an S[I]RL $\mathbf{A}$, its negative cone $A^- = \{a \in A : a \leq e\}$, can be turned into a Brouwerian algebra $\mathbf{A}^- = \langle A^-; \wedge, \vee, e, \to^- \rangle$, by restricting the operations $\wedge, \vee$ to A^- and defining

$$a \to^- b = (a \to b) \wedge e, \text{ for } a, b \in A^-.$$

For any Brouwerian algebra $\mathbf{B}$, let $\mathrm{Pr}(\mathbf{B})$ be the set of non-empty prime (lattice) filters F of $\mathbf{B}$, *including* B itself. We define the *depth* of F in $\mathrm{Pr}(\mathbf{B})$ to be the greatest $n \in \omega$ (if it exists) such that there is a chain in $\mathrm{Pr}(\mathbf{B})$ of the form $F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = B$. If no such n exists, we say F has *depth* ∞ in $\mathrm{Pr}(\mathbf{B})$. We define $\mathrm{depth}(\mathbf{B}) = \sup\{\mathrm{depth}(F) : F \in \mathrm{Pr}(\mathbf{B})\}$. If $\mathbf{A}$ is an S[I]RL and K is a variety of S[I]RLs, we define

$$\mathrm{depth}(\mathbf{A}) = \mathrm{depth}(\mathbf{A}^-) \quad \text{and} \quad \mathrm{depth}(\mathsf{K}) = \sup\{\mathrm{depth}(\mathbf{A}) : \mathbf{A} \in \mathsf{K}\}.$$

If a variety of S[I]RLs is finitely generated, then it has finite depth (but not conversely). Our main result is as follows.

Theorem 1. *Let K be a variety of S[I]RLs of finite depth such that each finitely subdirectly irreducible member of K is generated (as an algebra) by its negative cone. Then K has the ES property.*

The argument works also for S[I]RLs having distinguished least elements. Consequently, Theorem 1 generalizes the recent discovery in [1] that every variety of Brouwerian or Heyting algebras of finite depth has the ES property.

We can provide examples which show that neither of the two hypotheses in Theorem 1 can be dropped.

A *Sugihara monoid* is a De Morgan monoid that is idempotent (i.e., $x^2 = x$). It has recently been shown that every variety of Sugihara monoids has the ES property, and that the same applies to the involution-less subreducts of Sugihara monoids [1]. For the finitely generated varieties of this kind, the ES property could alternatively be deduced immediately from Theorem 1.

The (non-idempotent) De Morgan monoid on the chain $\neg((\neg e)^2) < e < \neg e < (\neg e)^2$, denoted by $\mathbf{C}_4$, generates one of just four minimal varieties of De Morgan monoids [5], and it is the only 0–generated algebra onto which finitely subdirectly irreducible De Morgan monoids may be mapped by non-injective homomorphisms [7]. There is a largest variety U of De Morgan monoids consisting of homomorphic *preimages* of $\mathbf{C}_4$, and in the subvariety lattice of U, the variety generated by $\mathbf{C}_4$ has just ten covers [6]. All ten of these varieties satisfy the hypotheses of Theorem 1 and therefore have the ES property.

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Epimorphisms, definability and cardinalities*

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1 Introduction

A class K of similar algebras is called a *prevariety* if it is closed under subalgebras, direct products and isomorphic images. This amounts to the demand that K be axiomatized by some class Ξ of *implications* $\&\{\alpha_i = \beta_i : i \in I\} \Longrightarrow \alpha = \beta$ (I a possibly infinite set) [2]. The claim that we cannot always find a *set* to play the role of Ξ is consistent with the set theory NBG (including choice). Its negation (i.e., the claim that sets suffice) is consistent with NBG if huge cardinals exist. These facts were established by Adámek [1].

'Bridge theorems' of abstract algebraic logic [5, 7, 8, 9] have the form

$$\vdash \text{ has logical property } P \text{ iff } \mathsf{K} \text{ has algebraic property } Q,$$

where $\vdash$ is an algebraizable logic and $\mathsf{K} = \mathsf{Alg}(\vdash)$ is its algebraic counterpart—in which case K is a prevariety, at least. Examples include connections between Beth definability properties and the surjectivity of epimorphisms. Beth properties ask that, whenever a set Γ of formal assertions about the variables $\vec{x}, \vec{z}$ defines $\vec{z}$ implicitly in terms of $\vec{x}$, then it does so explicitly as well.

Blok and Hoogland [4] showed that the straightforward *ES property*—i.e., the demand that *all* epimorphisms in K be surjective—corresponds to an *infinite* version of the Beth property, where no cardinal is assumed to bound the lengths of the sequences $\vec{x}, \vec{z}$, nor the size of Γ. When testing for implicit definability, we need to substitute expressions for the variables $\vec{z}$, and this may introduce fresh variables. For such reasons, in the general bridge theorem connecting the ES and infinite Beth properties, the logic $\vdash$ is formulated not as a single (substitution-invariant) deductive system but as a *family* of deductive systems, indexed by the sub*sets* of a *proper class* of variables, with a correspondingly wider substitution-invariance demand.

This is unavoidable, in view of Adámek's findings, but the immersion of proper classes in the *syntax* of formal systems jars psychologically with the spirit of formalization. Moreover, many familiar algebraizable logics are finitary, with only countably many connectives, and are formalized using a countable set of variables—as nothing more is required for their axiomatization. When working with such logics, one would much prefer a suitably *localized* version of

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the infinite Beth property that also presupposes only a *set* of variables, but is still provably equivalent to the unrestricted ES property for the algebraic counterpart. From such a result, one might expect a formally stronger conclusion when inferring definability results from an analysis of epimorphisms.

It seems, however, that the published literature of abstract algebraic logic contains no such bridge theorem, and our aim here is to fill this gap.

Our results cater for all logics that are *equivalential* in the sense of [7], so they encompass all algebraizable logics. Each [finitely] equivalential logic has a [finite] family $\Delta(x, y)$ of binary terms that simulates a bi-conditional connective, to the extent that the Lindenbaum-Tarski construction may be carried out in a recognizable manner. (The family is essentially unique, so we abbreviate $\Delta(x, y)$ as $x \leftrightarrow y$ below.) Accordingly, the natural semantics for an *arbitrary* equivalential logic $\vdash$ is its class $\mathsf{Mod}^*(\vdash)$ of *reduced matrix models*, which coincides essentially with $\mathsf{Alg}(\vdash)$ when $\vdash$ is algebraizable.

As matrices are algebras with a distinguished unary relation, our analysis of epimorphisms will attend not only to pure algebras, but also to first order structures, where atomic formulas take over the role of equations. (Homomorphisms between similar structures preserve all indicated relations, but are not assumed to reflect them.)

2 Epimorphisms

Consider a class K of similar structures, and a K–morphism h, i.e., a homomorphism between members of K. Recall that h is called a (K–) *epimorphism* provided that, for any two homomorphisms f, g from the co-domain of h to a single member of K, if $f \circ h = g \circ h$, then $f = g$. A substructure $\boldsymbol{D}$ of a structure $\boldsymbol{B} \in \mathsf{K}$ is said to be (K–) *epic* in $\boldsymbol{B}$ if each homomorphism from $\boldsymbol{B}$ to a member of K is determined by its restriction to $\boldsymbol{D}$. (This amounts to the demand that the inclusion map $\boldsymbol{D} \to \boldsymbol{B}$ be an epimorphism, provided that K is closed under substructures.) Of course, a K–morphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is an epimorphism iff $h[\boldsymbol{A}]$ is an epic substructure of $\boldsymbol{B}$, so K has the (unrestricted) *ES property* iff each of its members has no proper epic substructure.

Already for algebras, the connection between 'implicitly defined' constructs and epimorphisms was remarked on in the literature long ago (e.g., see Freyd [10, p. 93] and Isbell [11]), but it is characterized in a syntactically sharper manner in Theorem 3 of Campercholi's recent paper [6]. There, however, it is confined to classes closed under ultraproducts. We extend it to arbitrary classes below (where, as usual, structure $\boldsymbol{A}$ has universe A, etc.)

Theorem 1. *Let K be any class of similar structures, $\boldsymbol{A}$ a substructure of $\boldsymbol{B} \in \mathsf{K}$ and $Z \subseteq B \backslash A$, where $A \cup Z$ generates (the algebra reduct of) $\boldsymbol{B}$.*

Then $\boldsymbol{A}$ is K–epic in $\boldsymbol{B}$ iff, for each $b \in Z$, there is a set $\Sigma = \Sigma(\vec{x}, \vec{z}, v)$ of atomic formulas such that $\boldsymbol{B} \models \Sigma(\vec{a}, \vec{c}, b)$ for suitable $\vec{a} \in A$ and $\vec{c} \in B$, and

$$\mathsf{K} \models \&[\Sigma(\vec{x}, \vec{z}, v_1) \cup \Sigma(\vec{x}, \vec{y}, v_2)] \Longrightarrow v_1 = v_2.$$

In this case, for each $b \in Z$, we can arrange that $|\vec{z}| \leq |Z|$ and $\vec{c} \in Z$.

When K is closed under ultraproducts (in particular, when K is a *quasivariety*), then each of the sets Σ in Theorem 1 can be chosen finite.

In a structure $\boldsymbol{B}$, a substructure $\boldsymbol{A}$ is said to be *almost total* if $\boldsymbol{B}$ is generated by $A \cup Z$ for some *finite* $Z \subseteq B$. We say that K has the *weak ES property* if no $\boldsymbol{B} \in \mathsf{K}$ has a proper K–epic almost total substructure. The meaning of this demand would not change if, in the definition of 'almost total', we required $|Z| = 1$ (see [4, p. 76]).

The aforementioned findings of Adámek justify our interest in the following prevarieties.

Definition 2. For each infinite cardinal $\mathfrak{m}$, an $\mathfrak{m}$–*generalized quasivariety* is a class of structures axiomatized by implications, each of which is formulated in at most $\mathfrak{m}$ variables.

Theorem 3. *For $\mathfrak{m} \geqslant \aleph_0$, let K be an $\mathfrak{m}$–generalized quasivariety whose signature has cardinality $\mathfrak{s}$. Then K has the ES property iff no structure in K of cardinality at most $\mathfrak{m}+\mathfrak{s}$ has a proper K–epic substructure.*

The proof uses Theorem 1. As every quasivariety is an $\aleph_0$–generalized quasivariety, we infer:

Corollary 4. *Let K be a quasivariety with a countable signature. Then K has the ES property if and only if no countable member of K has a proper K–epic substructure.*

For *varieties* K of algebras, Corollary 4 follows from a finding of Isbell [11, Cor. 1.3], whose own proof relies, however, on closure under homomorphic images.

An $\aleph_0$–generalized quasivariety need not be a quasivariety: see [1, p. 45]. Nevertheless, in Corollary 4, we cannot strengthen 'countable' to 'finitely generated'. Indeed, a locally finite variety K of Brouwerian algebras and a proper K–epic subalgebra of a denumerable member of K are exhibited in [3, Sec. 6], but no finitely generated (i.e., finite) member of K has a proper K–epic subalgebra, because every variety of Brouwerian algebras has the *weak* ES property [12].

On the other hand, finitely generated structures do suffice, in quasivarieties, to test the weak ES property itself (again, owing to Theorem 1):

Theorem 5. *A quasivariety K has the weak ES property iff no finitely generated member of K has a proper K–epic substructure.*

3 Bridge Theorems

Below, the class of all sub*sets* of a class C is denoted by $\mathcal{P}(C)$. Given an algebraic signature $\mathcal{L}$ and a set X of variables, the absolutely free algebra of $\mathcal{L}$–terms over X is denoted by $\boldsymbol{T}(X)$.

Definition 6. ([4]) An equivalential logic $\vdash$ over a proper class Var of variables is said to have the *(deductive) infinite Beth (definability) property* if the following holds for all disjoint subsets X, Z of Var, with $T(X) \neq \emptyset$, and all $\Gamma \subseteq T(X \cup Z)$: if,

> for each $z \in Z$ and each homomorphism $h \colon \boldsymbol{T}(X \cup Z) \to \boldsymbol{T}(Y)$, with $Y \in \mathcal{P}(\mathit{Var})$, such that $h(x) = x$ for all $x \in X$, we have $\Gamma \cup h[\Gamma] \vdash z \leftrightarrow h(z)$,

then, for each $z \in Z$, there exists $\varphi_z \in T(X)$ such that $\Gamma \vdash z \leftrightarrow \varphi_z$.

Theorem 7. ([4, Thm. 3.12]) *Let $\vdash$ be an equivalential logic over a proper class. Then $\vdash$ has the infinite Beth property iff, in the prevariety $\mathsf{Mod}^*(\vdash)$, all epimorphisms are surjective.*

We have not found the following definition (and theorem) in the published literature.

Definition 8. An equivalential logic $\vdash$ over an infinite set V of variables will be said to have the (V–) *localized infinite Beth property* provided that the following is true for all disjoint subsets X, Z of V and all $\Gamma \subseteq T(X \cup Z)$, such that $T(X) \neq \emptyset$ *and* $|V \backslash (X \cup Z)| \geq |Z| + \aleph_0$: if,

> for each $z \in Z$ and each *endomorphism* h of $\boldsymbol{T}(V)$, such that $h(x) = x$ for all $x \in X$, we have $\Gamma \cup h[\Gamma] \vdash z \leftrightarrow h(z)$,

then, for each $z \in Z$, there exists $\varphi_z \in T(X)$ such that $\Gamma \vdash z \leftrightarrow \varphi_z$.

Theorem 9. *Let $\vdash$ be an equivalential logic over an infinite set V, where $\vdash$ has at most $|V|$ connectives and has the localized infinite Beth property. Then no member of* $\mathsf{Mod}^*(\vdash)$ *with at most $|V|$ elements has a proper* $\mathsf{Mod}^*(\vdash)$*–epic submatrix.*

This, with Theorems 3 and 7, facilitates the proof of a bridge theorem in which the need to consider proper classes is eliminated for symbolically limited logics:

Theorem 10. *Let $\vdash$ be an equivalential [resp. algebraizable] logic over an infinite set V, where $\mathfrak{s}$ is the cardinality of the signature. Assume that $\vdash$ has an axiomatization that uses at most $\mathfrak{m}$ variables, where $\mathfrak{m}+\mathfrak{s} \leq |V|$. Then $\vdash$ has the localized infinite Beth property iff all epimorphisms in* $\mathsf{Mod}^*(\vdash)$ *[resp.* $\mathsf{Alg}(\vdash)$*] are surjective.*

Nearly every 'familiar' logic $\vdash$ has countable type and either is algebraized by a quasivariety or is finitary and equivalential. For such a logic, the set V can be chosen denumerable and the assumption $\mathfrak{m}+\mathfrak{s} \leq |V|$ in Theorem 10 becomes redundant, but the need to 'localize' does not.

The *finite Beth property* is defined like the infinite one, except that the set Z in its definition is required to be finite. An equivalential logic $\vdash$ over a proper class has this property iff $\mathsf{Mod}^*(\vdash)$ has the weak ES property [4, Thm. 3.14, Cor. 3.15].

Let us also define the (V–) *localized finite Beth property* like its infinite analogue, but stipulating that Z be finite and substituting '$V \backslash X$ is infinite' for '$|V \backslash (X \cup Z)| \geq |Z| + \aleph_0$'.

Theorem 10 has analogues for these properties. We state only one, wherein the cardinality of the signature plays no role.

Theorem 11. *Let $\vdash$ be a logic over a denumerable set V of variables, where $\vdash$ is algebraized by a quasivariety [resp. is finitary and finitely equivalential]. Then $\vdash$ has the localized finite Beth property iff* $\mathsf{Alg}(\vdash)$ *[resp.* $\mathsf{Mod}^*(\vdash)$*] has the weak ES property.*

Observe that, when applying Theorem 11, we need only test the surjectivity of epimorphisms into finitely generated targets, because of Theorem 5. The equivalent conditions of Theorem 11 obtain for all super-intuitionistic logics (and the corresponding varieties of Heyting algebras).

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Residuated lattices and the Nelson identity

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In this talk I report on ongoing joint research with Matthew Spinks and Thiago Nascimento; our main focus is an investigation into the meaning and consequences of what we call the *Nelson identity* in the context of residuated lattices.

Nelson's constructive logic with strong negation **N3** [7, 12, 13, 16] can be viewed as either a conservative expansion of the negation-free fragment of intuitionistic logic by a new unary logical connective of *strong negation* or (to within definitional equivalence) as the axiomatic extension $\mathbf{NInFL}_{ew}$ of the involutive full Lambek calculus with exchange and weakening by the *Nelson axiom*:

$$\vdash ((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x))) \Rightarrow (x \Rightarrow y). \qquad (\text{Nelson}_{\vdash})$$

The algebraic counterpart of $\mathbf{NInFL}_{ew}$ is the recently introduced class NRL of Nelson residuated lattices, which is term equivalent to the variety of *Nelson algebras*, the traditional algebraic counterpart of **N3**. Members of NRL are *compatibly involutive commutative integral residuated lattices* satisfying the algebraic counterpart of the axiom ($\text{Nelson}_{\vdash}$):

$$((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x))) \Rightarrow (x \Rightarrow y) \approx \mathbf{1}. \qquad (1)$$

The starting point for the present contribution is our previous work devoted to a weaker logic of strong negation introduced by Nelson in [8] under the name of **S**. We established in [5, 6] that **S** is the axiomatic extension of the full Lambek calculus with exchange and weakening by the axioms of *double negation* and $(3, 2)$-*contraction*, *viz.*

$$\vdash {\sim}{\sim} x \Rightarrow x$$
$$\vdash (x \Rightarrow (x \Rightarrow (x \Rightarrow y))) \Rightarrow (x \Rightarrow (x \Rightarrow y)).$$

In view of results due to Spinks and Veroff [14, 15] and Busaniche and Cignoli [3], Nelson's logic **N3** is precisely the extension of **S** by the axiom ($\text{Nelson}_{\vdash}$).

We showed in [5, 6] that **S** is algebraisable and we characterised its algebraic counterpart as the variety of *compatibly involutive 3-potent commutative integral residuated lattices* (dubbed for short as **S**-*algebras*). In consequence, the algebraic counterpart of **N3** is, up to term equivalence, precisely the subvariety of **S**-algebras that additionally satisfies the identity (1) above. Busaniche and Cignoli [3, Remark 2.1] have observed that (1) is equivalent (over compatibly involutive commutative integral residuated lattices) to the following

$$(x \Rightarrow (x \Rightarrow y)) \wedge ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)) \approx x \Rightarrow y \qquad (\text{Nelson})$$

which we take as our official version of the *Nelson identity*.

The present contribution is the outgrowth of our interest in understanding the essential difference between the logics **S** and **N3** in a (universal) algebraic context; our main focus shall thus be on the meaning and role of the Nelson identity in the context of compatibly involutive commutative integral residuated lattices. In this endeavour, we were naturally led to formulate more abstract order-theoretic/algebraic properties which go hand in hand, in our context, with the Nelson identity. We have thus introduced generalisations of the *congruence orderable algebras* and the *Fregean varieties* of [4]. The main interest in our approach is, in our opinion, the fact that it may open the way to further universal algebraic investigation beyond the context of Nelson's logics and even beyond residuated lattices.

Within the framework of compatibly involutive residuated lattices, it is natural to work with two distinguished algebraic constants instead of just one; this led us to introduce the following generalisations of the universal algebraic notions of congruence orderable and Fregean algebra. Call an algebra $\mathbf{A}$ with residually distinct constants $\mathbf{0}^{\mathbf{A}}$ and $\mathbf{1}^{\mathbf{A}}$

- $(\mathbf{0},\mathbf{1})$-congruence orderable if for all $a,b \in A$, the following congruence condition holds:

$$\Theta^{\mathbf{A}}(\mathbf{0}^{\mathbf{A}},a)=\Theta^{\mathbf{A}}(\mathbf{0}^{\mathbf{A}},b) \ \text{ and } \ \Theta^{\mathbf{A}}(\mathbf{1}^{\mathbf{A}},a)=\Theta^{\mathbf{A}}(\mathbf{1}^{\mathbf{A}},b) \ \text{ implies } \ a=b;$$

- $(\mathbf{0},\mathbf{1})$-Fregean if $\mathbf{A}$ is $(\mathbf{0},\mathbf{1})$-congruence orderable and the congruences on $\mathbf{A}$ are uniquely determined by both their $\mathbf{0}^{\mathbf{A}}$- and $\mathbf{1}^{\mathbf{A}}$-equivalence classes simultaneously.

One of our main results is the following.

Theorem. For a compatibly involutive commutative integral bounded residuated lattice $\mathbf{A}$, the following are equivalent:

1. $\mathbf{A}$ *is a Nelson residuated lattice.*
2. $\mathbf{A}$ *is* $(\mathbf{0},\mathbf{1})$*-congruence orderable.*
3. $\mathbf{A}$ *is* $(\mathbf{0},\mathbf{1})$*-Fregean.*

As mentioned above, the property of being $(\mathbf{0},\mathbf{1})$-congruence orderable (or Fregean) is meaningful not only in the context of residuated lattices but in principle for any class of algebras having two distinguished constants. It is therefore possible as well as natural to ask questions such as:

- Is the class of $(\mathbf{0},\mathbf{1})$-congruence orderable (or Fregean) bounded (but not necessarily involutive) residuated lattices a variety? If so, what is an equational presentation for it?
- Is there a characterisation theorem for congruence permutable $(\mathbf{0},\mathbf{1})$-congruence orderable varieties similar to the characterisation theorem for congruence permutable congruence $\mathbf{1}$-orderable varieties of Idziak et al. [4]?

Even more generally, one could replace the constants $\mathbf{0}$ and $\mathbf{1}$ in the definition of $(\mathbf{0},\mathbf{1})$-congruence orderability with unary (not necessarily constant) terms $\mathbf{s}(x)$ and $\mathbf{t}(x)$ to investigate $(\mathbf{s},\mathbf{t})$-congruence orderable algebras; in this context, a natural condition that may make the study of $(\mathbf{s},\mathbf{t})$-congruence orderable algebras comparatively tractable would be to require $\mathbf{s}(x)$ and $\mathbf{t}(x)$ to satisfy τ-normality in the sense of [2, Definition 5.1.2]. This would allow us to extend our study, for instance, to the class of **N4**-lattices, the algebraic counterpart of *paraconsistent Nelson's logic* [1, 9, 10, 11], which do not have any term definable algebraic constants. We leave these as suggestions for potentially interesting directions of future research.

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Modal logics for reasoning about weighted graphs

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Summary. Frames for lattice-valued modal logics [3, 5, 4, 9] are directed weighted graphs (with the given lattice of truth values seen as an algebra of weights), so it is natural to consider these logics as formalisms for reasoning about weighted graphs; in a similar vein, versions of modal logic based on classical logic have been used to express and reason about properties of non-weighted graphs [2, 7].

In this position talk we examine existing many-valued modal logics as formalisms for reasoning about weighted graphs. In addition to presenting preliminary results, this examination will lead us to propose a number of topics for future research in many-valued modal logic. In the rest of the abstract, we give a more detailed overview of the contents of the talk.

Weighted graphs. Let $\boldsymbol{A}$ be a complete FL_{ew} algebra with bounds 0 and 1 (we use FL_{ew} algebras for the sake of generality, although in most examples we will use linearly ordered algebras over $[0,1]$). An $\boldsymbol{A}$-weighted graph is $\mathfrak{G} = \langle V, E \rangle$ where V is a non-empty set (of vertices) and E is a function from $V \times V$ to $\boldsymbol{A}$ (the edge function). Intuitively, $\boldsymbol{A}$ is seen as an algebra of weights with the fusion operation $\odot$ representing addition of weights; 1 is the minimal weight ("no weight") and 0 is the maximal possible ("infinite") weight. $E(v, v')$ is the weight of the edge (v, v')—informally, this can be seen as the "distance" between v and v' or as a representation of the cost of getting (directly) from v to v'. It might be a bit unintuitive at first to interpret 1 and 0 as the minimal and maximal weight, respectively, but the following picture might help. We may see "distances" or "costs" as something that reduces available "resources" also expressed in terms of $\boldsymbol{A}$—if a is the amount of available resources and b is a weight of an edge between v and v', then $a \odot b$ is the amount of resources left after getting from v to v' using the given edge. In particular, if the an edge has no weight, i.e. it's weight is 1, then $a \odot 1 = a$; this means that using the edge does not reduce the amount of available resources. Similarly, $a \odot 0 = 0$ so using an edge with "infinite" weight 0 does not leave any resources.

Modal languages. Let *Lab* be a set of labels. A labelled $\boldsymbol{A}$-weighted graph is $\langle \mathfrak{G}, L \rangle$ where $L \colon V \to \mathcal{P}(Lab)$ is a labelling function assigning to each vertex a set of labels. Formulas of the language $\mathcal{L}(Lab)$ are constructed from *Lab* (seen as propositional atoms) using the operators $\top, \bot$ (nullary), $\Diamond, \Box$ (unary) and $\wedge, \vee, \rightarrow, \odot$ (binary). For each $v \in V$ and L we define a function v_L from the formulas of $\mathcal{L}(Lab)$ to $\boldsymbol{A}$ as follows (sometimes denoted as v):

$v_L(\top) = 1$ and $v_L(\bot) = 0$

$v_L(p) = 1$ if $p \in L(v)$ and $v_L(p) = 0$ otherwise (for $p \in Lab$)

$v_L(\varphi \circ \psi) = v_L(\varphi) \circ v_L(\psi)$ for $\circ \in \{\wedge, \vee, \odot, \rightarrow\}$

$v_L(\Diamond\varphi) = \bigvee_{u \in V}\{E(v, u) \odot u_L(\varphi)\}$

$v_L(\Box\varphi) = \bigwedge_{u \in V}\{E(v, u) \rightarrow u_L(\varphi)\}$

Examples of expressiveness. It is easy to see that $v(\Diamond p)$ is the smallest distance from v to a vertex labelled with p, if the underlying algebra of weights is linear. The reason is that $u_L(p) \in \{0, 1\}$ and so $E(v, u) \odot u(p)$ is either 0 (if u is not labelled by p) or $E(v, u)$ (if u is labelled by p). Hence, $v(\Diamond p)$ is the supremum of the weights of edges connecting v with vertices labelled by p. On the reading of 1 as the smallest possible distance, the supremum of weights can be read as the infimum of distances; if the algebra is linearly ordered, then infimum is minimum. (The interpretation of $\Diamond p$ in non-linear weight algebras is less straightforward; one may see this as a reason to work with linear algebras only.)

Let us define $\Diamond^1\varphi := \Diamond\varphi$ and $\Diamond^{k+1}\varphi := \Diamond\Diamond^k\varphi$ for $k \geq 1$. We can then interpret $v(\Diamond^k p)$ as the weight of the "lightest" ("most feasible", "cheapest", "shortest") k-member path to a vertex labelled with p starting at v (a k-member path starting at v is a function from $\{0, 1, \ldots, k\}$ to V where the value of 0 is v). As a special case, $v(\Diamond^k\top)$ is the weight of the lightest k-member path starting at v.

Similarly, $v(\Diamond^k p \to \Diamond^m q) = 1$ iff the lightest k-member path to (a vertex labelled by) p is not cheaper than the lightest m-member path to q (as a result, $\Diamond\Diamond p \to \Diamond p$ allows us to define a weak version of transitivity—the direct path from v to u is never more expensive than a path via some w). Introducing constants (for elements in $\boldsymbol{A}$) into the language allows us to have formulas establishing that the cost of the most feasible k-member path to p is smaller than (bigger than, equal to) a specific constant-denoted weight. For instance, $v(\bar{a} \to \Diamond^k p) = 1$ iff $a \leq v(\Diamond^k p)$ so, in a sense, $\bar{a} \to \Diamond^k p$ "says" that the cost of the most feasible k-member path to p is not bigger than a.

Axiomatization. By defining $\mathcal{L}$ to be the local consequence relation over labeled $\boldsymbol{A}$-weighted graphs in the standard manner, we obtain a logic that has a number of interesting properties (for example, it is not structural, i.e. not closed under arbitrary substitutions). For the cases where an axiomatization of the modal logic over Kripke models with valued accessibility relations is known (e.g. [3],[4]), it is easy to provide an axiomatic system (not substitution invariant) for $\mathcal{L}$, simply combining classical propositional logic closed under substitution of only non-modal formulas, and the usual axiomatization of the modal logic.

Extensions of the basic framework. As noted already in [2], versions of the basic modal language fail to express most of the interesting graph properties. This is also the case in the weighted setting. One extension to consider is a many-valued version of hybrid logic [1]. The literature on these matters is scarce; [6] consider hybrid logics based on finite and infinite standard Gödel algebras and [8] study hybrid logics over arbitrary finite Heyting algebras. One issue raised by the present considerations is a generalization of these approaches to a setting of arbitrary FL_{ew} algebras.

A hybrid language $\mathcal{L}(Lab, Nom)$ adds to $\mathcal{L}(Lab)$ a set Nom, disjoint from Lab, of expressions called *nominals*. Formulas of the language are built from Lab and Nom as before with the addition of unary operators $@_i$ for $i \in Nom$. The labelling function is now defined as $L : V \to 2^{Lab} \times Nom$ such that L is injective on Nom (each vertex is labelled by precisely one nominal). We denote as $d(i)$ the unique vertex v such that $i \in L(v)$. The definition of v_L is extended by

$$v_L(i) = 1 \text{ if } v = d(i) \text{ and } v_L(i) = 0 \text{ otherwise (for all } i \in Nom)$$

$$v_L(@_i\varphi) = d(i)_L(\varphi)$$

It can be shown that formulas of the hybrid language can define acyclic graphs ($@_i(\Diamond i \to \bot)$ if the underlying graph is transitive). In fact, the "combined weight" of each finite path $\langle d(i_1), \ldots, d(i_n)\rangle$ corresponds to the value of the formula $@_{i_1}\Diamond(i_2 \odot \Diamond(\ldots i_{n-1} \odot \Diamond i_n)\ldots)$.

In addition, it can be shown that each finite weighted graph can be characterized, up to isomorphism, by a formula of a many-valued hybrid language with a finite number of constants. A study of general many-valued hybrid logics (including axiomatization) is a topic of current research.

There is a number of further extensions of the framework that offer expressive power useful in the context of weighted graphs. For instance, extend the language by an unary operator σ on nominals with the truth condition

$$v_L(\sigma(i)) = \bigodot_{v\in V}\{E(d(i), v) \mid E(d(i), v) \neq 0\}$$

In other words, $\sigma(i)$ is the combined weight of the edges coming out of the vertex $d(i)$ (in the fashion of the well-established *magic labelling* [10] for weighted graphs). This operator does not have a straightforward counterpart in the classical hybrid setting. The study of many-valued hybrid logic with σ is a topic of current research.

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Deciding active structural completeness

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The notions of admissibility and various types of structural completeness for consequence relations (considered as sets of rules) and logics (considered as sets of formulas closed under some *default* rules) has received attention for many years. In this talk, we focus on straightforward algebraic analogs. Our results apply to strongly algebraziable consequence relations (logics).

A *quasivariety* $\mathcal{Q}$ is a class of algebras defined by quasi-identities, i.e., by formulas of the form $\bigwedge s_i(\bar{x}) \approx t_i(\bar{x}) \to s(\bar{x}) \approx t(\bar{x})$. A *variety* is a class defined by identities, i.e., by formulas of the form $s(\bar{x}) \approx t(\bar{x})$. A quasi-identity $q = \bigwedge s_i(\bar{x}) \approx t_i(\bar{x}) \to s(\bar{x}) \approx t(\bar{x})$ is *admissible* for a quasivariety $\mathcal{Q}$ provided for every tuple $\bar{u}$ of terms if $\mathcal{Q} \models \bigwedge s_i(\bar{u}) \approx t_i(\bar{u})$ then $\mathcal{Q} \models s(\bar{u}) \approx t(\bar{u})$. This condition says that q holds in free algebras for $\mathcal{Q}$. And q is *active* for $\mathcal{Q}$ if there exists $\bar{u}$ such that $\mathcal{Q} \models \bigwedge s_i(\bar{u}) \approx t_i(\bar{u})$. This is equivalent to the satisfiability of $\bigwedge s_i(\bar{u}) \approx t_i(\bar{u})$ in any free algebra for $\mathcal{Q}$. We say that $\mathcal{Q}$ is (*actively*[1]) *structurally complete* [(A)SC for short] if every (active) admissible for $\mathcal{Q}$ quasi-identity holds in $\mathcal{Q}$.

We are interested in decidability of (A)SC. In order to make the problem meaningful a (quasi)variety should be given in a finitary way (we assume that the language of considered algebras is finite). There are two basic ways to do this: by giving a finite axiomatization or by giving a finite generating set of finite algebras. We undertake the later approach. Here is the main problem.

Problem 1. Is it decidable whether the variety generated by a given finite algebra is (A)SC?

The parallel problem for quasivarieties was solved by Dywan and Bergman (SC) and by Metcalfe and Röthlisberger (ASC).

Theorem 2 ([1, 2, 6])**.** *It is decidable whether a finite set $\mathcal{G}$ of algebras in a finite language generates the quasivariety wich is (A)SC.*

As this fact is important also for varieties, let us look at it a bit closer. Interestingly, these solutions are based on different algorithms. Dywan's solution is based on the observation that if k bounds the cardinality of members of $\mathcal{G}$, then for SC it is enough to check only quasi-identities with at most k variables. (His technique works as well for ASC but one has to check quasi-identities with at most $k \cdot k^k$ variables.) Bergman's (for SC) and Metcalfe's and Röthlisberger's (for ASC) solutions are based on studying generating sets for quasivarieties, in particular on relatively subdirectly irreducible algebras.

Theorem 3 ([1, 3, 6])**.** *Let $\mathcal{Q}$ be a quasivariety, $\mathbf{F}$ a free algebra for $\mathcal{Q}$ of denumerable rank, $\mathbf{M}$ its (as small as possible) subalgebra, and $\mathsf{Q}(\mathbf{F})$ be the quasivariety generated by $\mathbf{F}$. Then*

1. *$\mathcal{Q}$ is SC if and only if every relatively subdirectly irreducible algebra in $\mathcal{Q}$ belongs to $\mathsf{Q}(\mathbf{F})$;*
2. *$\mathcal{Q}$ is ASC if and only for every every relatively subdirectly irreducible algebra $\mathbf{S}$ in $\mathcal{Q}$ the algebra $\mathbf{S} \times \mathbf{M}$ belongs to $\mathsf{Q}(\mathbf{F})$.*

Now if $\mathcal{Q}$ is generated by a finite set $\mathcal{G}$ of finite algebras of cardinality at most k, then

[1] In most papers the adjective *almost* is used.

- $\mathsf{Q}(\mathbf{F}) = \mathsf{Q}(\mathbf{F}(k))$, where $\mathbf{F}(k)$ is a free algebra for $\mathcal{Q}$ of rank k;
- all relatively subdirectly irreducible algebras in $\mathcal{Q}$ embed into members of $\mathcal{G}$;
- $\mathbf{F}(k)$ and $\mathbf{M}$ are finite: $|F(k)| \leqslant a^{a^k}$ and $|M| \leqslant |F(1)| \leqslant a^a$, where $a = k^{|\mathcal{G}|}$.

Thus, indeed Theorem 3 gives an alogrithm for deciding (A)SC for finitely generated quasivarieties. The situation for varieties is more complicated. Still, the above considerations give algorithms for checking (A)SC for finitely generated varieties with finite computable bound on the cardinality of subdirectly irreducible algebras. Such varieties are finitely generated as quasivarieties. In particular, by Jónsson's lemma, we get the following fact.

Corollary 4. *It is decidable whether the congruence distributive variety given by a finite generating algebra is (A)SC.*

In [1] Bergman observed that if the variety $\mathcal{V}$ generated by a finite algebra $\mathbf{A}$ is SC, then every subdirectly irreducible algebra in $\mathcal{V}$ embeds into $\mathbf{A}$. Thus for finitely generated SC varieties there is a finite computable bound on the cardinality of subdirectly irreducible algebras. Our main contribution is the following analogue for ASC.

Theorem 5. *Let $\mathcal{V}$ be the ASC variety generated by an algebra $\mathbf{A}$. If $\mathbf{S}$ is subdirectly irreducible algebra in $\mathcal{V}$, then $|S| \leqslant |A|^{(|A|+1)\cdot|A|^{2\cdot|A|}}$.*

By Bergman's and our results and by facts about subdirectly irreducible algebras in congruence modular varieties obtained by Freese and McKenzie in [4], we can extend Corollary 4.

Corollary 6. *It is decidable whether the congruence modular variery given by a finite generating algebra is (A)SC.*

Let us finish with the following problem.

Problem 7. Is it decidable whether a finite algebra $\mathbf{A}$ generates a variety in which all subdirectly irreducible members have cardinality bounded by $|A|^{(|A|+1)\cdot|A|^{2\cdot|A|}}$?

If Problem 7 has a positive solution, then our main Problem 1 has a positive solution too. Note however that, in general, it is undecidable whether a finite algebra generates a variety in which all subdirectly irreducible members are finite [5].

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Non axiomatizability of the finitary Łukasiewicz modal logic

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Many-valued modal logics, understood as the logics arising from Kripke models evaluated over residuated lattices, is a field under development in which many questions remain open. In particular, while their semantic definition is clear, formulating corresponding axiomatic systems for these logics is one of the problems under study (see eg. [3, 4], [1], [8, 7], [10, 9]). We focus in this work on a problem arising from [8, 7] concerning the modal logic of the Kripke frames with a crisp accessibility relation evaluated over the standard Łukasiewicz algebra $[0,1]_{\text{Ł}}$. In particular, in the previous works, a Hilbert style axiomatic system is defined that is strongly complete with respect to the usual relational semantics (i.e., complete also for infinite sets of premises). This system, however, needs an infinitary inference rule (being naturally an infinitary logic), and the question whether its finitary companion can be axiomatized (meaning, as usual, axiomatized by a R.E set of rules), as can be done at a propositional level, remained open.

We can prove that such an axiomatization cannot exist for the global modal logic, by the combination of three properties: undecidability of the global consequence over finite models of the class, decidability of the propositional logic and completeness of the global consequence (over arbitrary models) with respect to so-called witnessed models. In what follows we show a sketch of the development of the above claim.

Let us remark that, in contrast to the more classical cases, where sometimes modal logics are identified with their theories (and so, sets of formulas), in this more general context we need to truly refer to the logic as a consequence relation. A ***logic*** $\mathcal{L}$ is a substitution invariant consequence relation on the set of formulas. As usual, a logic $\mathcal{L}$ is ***finitary*** whenever

$$\langle \Gamma, \varphi\rangle \in \mathcal{L} \text{ if and only if } \exists \Delta \subseteq_{\text{fin}} \Gamma \text{ s.t } \langle \Delta, \varphi\rangle \in \mathcal{L}$$

Consequently, the ***finitary companion*** of a logic $\mathcal{L}$, denoted by $\mathcal{L}^{\text{fin}}$ (and which can be proven to be a logic too) is defined as the set

$$\{\langle \Gamma, \varphi\rangle \colon \exists \Delta \subseteq_{\text{fin}} \Gamma \text{ s.t } \langle \Delta, \varphi\rangle \in \mathcal{L}\}$$

By well known Craig's lemma, we say that a logic $\mathcal{L}$ is ***axiomatizable*** when its set of consequence relations is Recursively Enumerable. This coincides with the idea of characterizing a logic axiomatized by a set of inference rules $\mathcal{R}$ as the minnimal one containing all rules in $\mathcal{R}$.

As usual, let us consider a language including $\Box$ and $\Diamond$ modal operators. A ***frame*** is a pair $\langle W, R\rangle$ where W is a set of so called worlds and $R \subseteq W \times W$ (we will write Rvw instead of $\langle v, w\rangle \in R$). We say that $\mathfrak{M} = \langle W, R, e\rangle$ is a (standard) ***Łukasiewicz Kripke model*** whenever $\langle W, R\rangle$ is a frame and $e \colon Fm \times W \to [0,1]_{\text{Ł}}$ such that it is (world-wise) a propositional homomorphism and moreover

$$e(\Box\varphi, v) = \bigwedge_{Rvw} e(\varphi, w) \quad \text{and} \quad e(\Diamond\varphi, v) = \bigvee_{Rvw} e(\varphi, w)$$

We let $\mathbb{K}$ be the class of all Łukasiewicz Kripke models, and $\mathbb{K}^{\omega}$ the class of all finite ones (i.e., where W is a finite set).

Fixing a class of Łukasiewicz Kripke models $\mathbb{C}$, we let $\mathcal{L}_{\mathbb{C}}$[1] be the logic given by the pairs $\langle \Gamma, \varphi\rangle$ such that, for any $\mathfrak{M} \in \mathbb{C}$,

$$[\forall w \in W, \forall \gamma \in \Gamma \quad e(\gamma, w) = 1] \Longrightarrow [\forall w \in W \quad e(\varphi, w) = 1].$$

Analogously, given a frame $\mathfrak{F}$, we let $\mathcal{L}_{\mathfrak{F}}$ be the logic given by the pairs $\langle \Gamma, \varphi\rangle$ such that $\langle \Gamma, \varphi\rangle \in \mathcal{L}_{\mathbb{C}}$, for $\mathbb{C}$ being the collection of all Łukasiewicz Kripke models whose underlying frame is $\mathfrak{F}$.

In what follows, we assume $\Gamma \cup \{\varphi\}$ to be some arbitrary finite set of formulas.

To begin, the following is a particular case of the undecidability results presented in [11], that contrast with the decidability results shown for local Gödel modal logics [2].

Lemma 2.1. *It is undecidable whether* $\langle \Gamma, \varphi\rangle \in \mathcal{L}^{fin}_{\mathbb{K}^\omega}$.

It is not hard to show the following (it also holds more in general, as long as the underlying propositional logic is decidable).

Lemma 2.2. *Let $\mathfrak{F}$ be a finite frame. Then it is decidable whether* $\langle \Gamma, \varphi\rangle \in \mathcal{L}_{\mathfrak{F}}$.

The idea of the proof is that, since $\mathfrak{F}$ is finite, we can define a suitable translation from global consequence relation over that particular frame to the Ł propositional logic over an extended set of propositional variables, that are associated to the worlds of $\mathfrak{F}$ (finitely many!). Since the latter is known to be decidable, so will be the consequence over $\mathfrak{F}$.

As a corollary, this implies that the problem of determining whether φ follows from Γ in all Łukasiewicz models of fixed cardinality is decidable. This corollary allows us to show that the set $\{\langle \Gamma, \varphi\rangle \in \mathcal{P}_{\text{fin}}(Fm) \times Fm \colon \langle \Gamma, \varphi\rangle \notin \mathcal{L}_{\mathbb{K}^\omega}\}$ is recursively enumerable. A procedure is that of enumerating all finite sets of formulas, and testing them against frames of increasing cardinality, which is decidable due to the previous corollary.

Together with the undecidability result stated in the beginning, this leads to the following non-axiomatizability result:

Corollary 2.3. $\mathcal{L}^{fin}_{\mathbb{K}^\omega}$ *is not axiomatizable.*

Observe, if it were to be so, we could easily enumerate $\{\langle \Gamma, \varphi\rangle \in \mathcal{P}_{\text{fin}}(Fm) \times Fm \colon \langle \Gamma, \varphi\rangle \in \mathcal{L}_{\mathbb{K}^\omega}\}$. Since from above we know that the set $\{\langle \Gamma, \varphi\rangle \in \mathcal{P}_{\text{fin}}(Fm) \times Fm \colon \langle \Gamma, \varphi\rangle \notin \mathcal{L}_{\mathbb{K}^\omega}\}$ is recursively enumerable, we would get that there is a decision procedure for $\mathcal{L}^{\text{fin}}_{\mathbb{K}^\omega}$, contradicting Lemma 2.1.

The next phase consists in seeing that, if there existed an axiomatization for $\mathcal{L}^{\text{fin}}_{\mathbb{K}}$, then there would also be a axiomatization for $\mathcal{L}^{\text{fin}}_{\mathbb{K}^\omega}$, contradicting the previous lemma. From [6, Lemma 3] we can prove that $\mathcal{L}_{\mathbb{K}}$ is complete with respect to *witnessed models*[2], i.e., $\mathcal{L}_{\mathbb{K}} = \mathcal{L}_{\text{wit}\mathbb{K}}$, for

$$\text{wit}\mathbb{K} \coloneqq \{\mathfrak{M} \in \mathbb{K} \colon \forall w \in W, \forall \varphi \in Fm, \exists w_{\Box_\varphi}, w_{\Box_\varphi} \in W \text{ s.t } \begin{cases} e(w, \Box\varphi) = & e(w_{\Box_\varphi}, \varphi), \text{ and} \\ e(w, \Diamond\varphi) = & e(w_{\Diamond_\varphi}, \varphi) \end{cases}\}$$

A characterization of the fragment over finite models in terms of the general $\mathcal{L}^{\text{fin}}_{\mathbb{K}}$ can be now given using some particular formulas for controlling the depth of the models.

Lemma 2.4. $\langle \Gamma, \varphi\rangle \in \mathcal{L}^{fin}_{\mathbb{K}^\omega}$ *if and only if for arbitrary* $p, q \notin \mathcal{V}ar(\Gamma, \varphi)$ *it holds* $\langle \Gamma \cup \Upsilon(p, q), \varphi \vee \Psi(p, q)\rangle \in \mathcal{L}^{fin}_{\mathbb{K}}$, *where*

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[1] This is usually named the global consequence over the class of models $\mathbb{C}$. We lighten the notation for a better reading in this abstract, since this notion of modal logic is the main one in this work.

[2] Observe that in the translation from global modal logic to predicate logic, the formulas in the resulting premises and consequence are all closed, thus meeting the premises of the referenced lemma.

- $\Upsilon(p,q) \coloneqq \{\Box 0 \vee (p \leftrightarrow \Box p), \Box 0 \vee (\Box p \leftrightarrow \Diamond p), (q \leftrightarrow p \,\&\, \Box q)\}$
- $\Psi(p,q) \coloneqq p \vee \neg p \vee q \vee \neg q$.

Archimedeanicity of $[0,1]_{\text{Ł}}$, together with completeness with respect to witnessed models allows us to get completeness of the deductions with the above structure with respect to finite models.

All the previous allow us to conclude our desired result:

Theorem 3.1. *$\mathcal{L}_{\mathbb{K}}^{fin}$ is not axiomatizable.*

Observe if it were, we could enumerate it, and through the previous lemma, also enumerate $\mathcal{L}^{\text{fin}}_{\mathbb{K}^{\omega}_{\text{Ł}}}$, contradicting the non-axiomatizability of this logic (Corollary 2.3).

Related questions & works

- It looks plausible that also the finitary companion of the so called local modal Łukasiewicz logic (where the consequence relation is defined by preserving truth at each world of the model) is also not axiomatizable. To prove this, it would be enough to know whether the extension of a local modal logic with the necessitation rule $\varphi \vdash \Box\varphi$ coincides with the corresponding global logic. This seems, however, not immediate, given the intrinsic infinitary character of the semantically-defined logic.
- The analogous result to the one detailled here can be proven for Modal Product Logic. A similar approach can be followed up to Corollary 2.3. However, it is not possible to proceed equally afterwards (it is not known whether product modal logic is complete with respect to some "controllable" class of models, only results up to validity have been proven [5]). Nevertheless, given that the 1-generated product subalgebras of $[0,1]_{\Pi}$ are still in the scope of the genneral formulation of Lemma 2.1 ([11]), it is possible to define a more involved translation showing that the global modal logic over $[0,1]_{\Pi}$-valued Kripke models cannot be axiomatized either, closing the main open problem from [10]. We will sketch this construction in the conference.

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Implicational tonoid logics and their relational semantics

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Abstract

This paper combines two classes of generalized logics, one of which is the class of *weakly implicative logics* introduced by Cintula and the other of which is the class of *gaggle logics* introduced by Dunn. For this purpose, we introduce implicational tonoid logics. First, we define several implicational tonoid logics in general and provide their algebraic semantics. Next, we introduce relational semantics, called Routley-Meyer-style semantics, for those implicational tonoid logics.

1 Introduction

One important trend in alternative logics is to introduce abstract logics with more general structures. In this setting, the logical connective "implication" is very important because systems of logic are often distinguished by the properties of their implications. In this paper, we focus on two classes of generalized logics, one is the class of *weakly implicative logics* introduced by Cintula, and the other is the class of *gaggle logics* introduced by Dunn. We will explore a combination of these two approaches, and for this purpose we introduce *implicational tonoid logics*. These logics can be regarded as both weakly implicative and partial gaggle-based logics.

Implications typically all share at least the underlying properties of reflexivity (φ implies φ is provable) and transitivity (if φ implies ψ and ψ implies χ are both provable, then φ implies χ is also provable). This suggests an abstraction based on preordered sets $(A, \preccurlyeq)$. But when A is an algebra, i.e., is outfitted with various operations of various degrees, we want an equivalence $\equiv$ to be a congruence as well. Preordered sets can be regarded as partially ordered sets by defining an equivalence relation $\equiv$ so that $a \equiv b$ iff $a \preccurlyeq b$ and $b \preccurlyeq a$.

Rasiowa clearly had this in mind when she [5] introduced the idea of an *implicative* algebra as a structure $(A, \mathsf{V}, \rightarrow)$, satisfying: 1. $a \rightarrow a = \mathsf{V}$; 2. if $a \rightarrow b = \mathsf{V}$ and $b \rightarrow c = \mathsf{V}$, then $a \rightarrow c = \mathsf{V}$; 3. if $a \rightarrow b = \mathsf{V}$ and $b \rightarrow a = \mathsf{V}$, then $a = b$; 4. $a \rightarrow \mathsf{V} = \mathsf{V}$. The motivating idea is that if we define $a \leq b$ iff $a \rightarrow b = \mathsf{V}$ then we get a partial-order. But it had a rather narrow scope compared to what it might cover now since Rasiowa's approach depends on the algebra having a greatest element V. This is appropriate for classical logic and intuitionistic logic.

But by the turn of the last millennium several logicians have investigated much weaker logics which do not share these properties. One of these is P. Cintula. He [1] introduced *weakly implicative logics* as a generalization of the implicative logics of Rasiowa [5]. Cintula generalizes the idea of an implicative algebra to in effect an "implicative matrix." A matrix $(A, \rightarrow, D)$ from an algebra in having a "designated" subset of elements $D \subseteq A$. This allows Cintula to define a preorder relation $\preccurlyeq$ based on the implication operation $\rightarrow$ by setting for any pair of elements a, b in $\mathcal{A}$, $a \preccurlyeq b$ if and only if $a \rightarrow b \in D$ (see Cintula & Noguera [2], pp. 104–105).

Another approach to implicative logics is due to one of the authors of this paper (Dunn). He [3] introduced *gaggles* as the acronym for "generalized Galois logics," which are a class of algebraic structures providing a unified approach to the semantics of non-classical logics. Originally the underlying structures were required to be distributive lattices, but he soon generalized this to include lattices and partially-ordered sets. In particular, he introduced *partial* gaggles as a name for gaggles with an underlying partial order.[1] Gaggles are based on a vast generalization of the well-known algebraic notion of a *Galois connection*, and includes *residuation* as a special case which allows one to define an implication, actually two implications, using: $b \leq a \to c$ iff $a \circ b \leq c$ iff $a \leq c \leftarrow b$[2] (*Residuation*). However it is clearly possible to ignore the operation $\circ$ and have just $b \leq a \to c$ iff $a \leq c \leftarrow b$.

Partial gaggles were first introduced by Dunn [4], and *tonoids* were also introduced there as a weakening of gaggles, even of partial gaggles, so as to replace residuation with the requirement that operations preserve, or co-preserve, the partial-order, and allowing this to vary from place to place. But even though implication was discussed in the context of both partial gaggles and tonoids, implicational tonoids were not explicitly introduced. This gap explains in large part why we have written the present paper.

2 Implicational tonoid logics: general cases

Here, we briefly introduce some important definitions and related results.

Definition 1 (Implicational tonoid logic)**.** Let $\mathcal{L}$ be a propositional language, such that $(\Rightarrow; 2) \in \mathcal{L}$ and let L be a logic in $\mathcal{L}$. We say that L is an *implicational tonoid logic* iff the following consecutions are elements of L: (R, reflexivity) $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$; ($MP$, modus ponens) $\varphi \Rightarrow \psi, \varphi \vdash_{\mathrm{L}} \psi$; ($T$, transitivity) $\varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$; ($Ton^i_{\#}$, tonicity) $\varphi \Rightarrow \psi \vdash_{\mathrm{L}} \#(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \Rightarrow \#(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n)$ or $\#(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n) \Rightarrow \#(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n)$ for each $(\#, n) \in \mathcal{L}$ and each $i \leq n$.

Let $\#$ be an n-ary connective. By $\#_n(\vec{\varphi}, \psi_i)$, we denote the application of $\#$ to n arguments, where $\vec{\varphi}$ is a sequence of $n-1$ elements of **VAR** and $\psi \in$ **VAR** is the i-th argument of $\#$, and similarly for $\#_n(\vec{\varphi}, \psi_i, \chi_j)$. We recall the definition of a *weakly implicative logic*. This will give us a chance to reveal an interesting relationship between tonoid logics and weakly implicative logics.

Definition 2 (Cintula (& Noguera) ([1, 2]), Weakly implicative logic)**.** L is said to be a *weakly implicative logic* iff (R), (MP), (T), and the following consecution are elements of L: for each $(\#, n)$, a part of $\mathcal{L}$, and each $i \leq n$, ($sCng^i_{\#}$, symmetrized congruence) $\varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \#_n(\vec{\chi}, \varphi_i) \Rightarrow \#_n(\vec{\chi}, \psi_i)$.

Theorem 3. *Implicational tonoid logics are weakly implicative logics.*

We then define some specific but still abstract implicational tonoid logics with each generalization of all the Galois and dual Galois connections and residuation and dual residuation.

Definition 4.

1. (Implicational partial Gaggle logic) L is said to be an *implicational partial Gaggle logic* if it is an implicational tonoid logic satisfying any of the following: for each $(f, n), (g, n) \in \mathcal{L}$

[1]It should be emphasized that these are not *partially-ordered algebras* in the standard sense which requires that the operations are preserved under the order. These are generalizations of this idea since each operation either preserves or inverts the order, and that can vary from place to place in the same operation. Thus e.g., $a \leq b$ implies $b \to c \leq a \to c$ and $c \to a \leq c \to b$. This led to the idea of a *tonoid* and the tonicity types of its elements. Cintula's weakly implicative logics do not have this property built into them.

[2]The forward arrow and the backward arrow are respectively the *right* and *left residuals*. The last is sometimes written as $b \rightsquigarrow c$, or something similar to avoid reversing the order.

and each $i \leq n$, (GC, Galois connected contraposition) $\varphi \Rightarrow f_n(\vec{\chi}, \psi_i) \dashv\vdash_{\mathrm{L}} \psi \Rightarrow g_n(\vec{\chi}, \varphi_i)$; ($dGC$, dual GC) $f_n(\vec{\chi}, \varphi_i) \Rightarrow \psi \dashv\vdash_{\mathrm{L}} g_n(\vec{\chi}, \psi_i) \Rightarrow \varphi$; ($RC$, residuated contraposition) $f_n(\vec{\chi}, \varphi_i) \Rightarrow \psi \dashv\vdash_{\mathrm{L}} \varphi \Rightarrow g_n(\vec{\chi}, \psi_i)$; ($dRC$, dual RC) $\varphi \Rightarrow f_n(\vec{\chi}, \psi_i) \dashv\vdash_{\mathrm{L}} g_n(\vec{\chi}, \varphi_i) \Rightarrow \psi$, where the tonicity types of f and g are the same and, in particular, their tonicity type of i-th arguments is antitone in each (GC) and (dGC). The tonicity types of f and g are different from each other in an argument distinct from i and, in particular, their tonicity type of i-th arguments is isotone in each (RC) and (dRC).

2. (Implicational residuated partial Gaggle logic) L is said to be an *implicational residuated partial Gaggle logic* if it is an implicational tonoid logic satisfying: for each $(f, n), (g, n), (h, n) \in \mathcal{L}$ and each $i, j \leq n$, (RES, residuation) $f_n(\vec{\chi}, \varphi_i, \psi_j) \Rightarrow \delta \dashv\vdash_{\mathrm{L}} \psi \Rightarrow g_n(\vec{\chi}, \varphi_i, \delta_j) \dashv\vdash_{\mathrm{L}} \varphi \Rightarrow h_n(\vec{\chi}, \psi_i, \delta_j)$, where the tonicity types of g and h are the same and the tonicity types of f and g (h resp.) are different from each other in an argument distinct from j. In particular, the tonicity types of f are isotone in its i-th and j-th arguments and the tonicity types of each g and h are antitone in its i-th argument and isotone in its j-th argument.

3. (Implicational dual residuated partial Gaggle logic) L is said to be an *implicational dual residuated partial Gaggle logic* if it is an implicational tonoid logic satisfying: for each $(f, n), (g, n), (h, n) \in \mathcal{L}$ and each $i, j \leq n$, ($dRES$, dual residuation) $\delta \Rightarrow f_n(\vec{\chi}, \varphi_i, \psi_j) \dashv\vdash_{\mathrm{L}} g_n(\vec{\chi}, \delta_i, \varphi_j) \Rightarrow \psi \dashv\vdash_{\mathrm{L}} h_n(\vec{\chi}, \delta_i, \psi_j) \Rightarrow \varphi$, where the tonicity types of g and h are the same and the tonicity types of f and g (h resp.) are different from each other in an argument distinct from i. In particular, the tonicity types of f are isotone in its i-th and j-th arguments and the tonicity types of each g and h are antitone in its j-th argument and isotone in its i-th argument.

Theorem 5 (Strong completeness)**.** L*, an implicational tonoid logic, is strongly complete w.r.t.* L*-matrices.*

Corollary 6. L*, an implicational (residuated, dual residuated) partial Gaggle logic, is strongly complete w.r.t.* L*-matrices.*

Moreover, we can introduce relational semantics called Routley-Meyer-style semantics and prove completeness.

Definition 7.

1. (Implicational Routley-Meyer-style (R-M) frame) For an implicational partially ordered set matrix $\mathcal{A} = (A, \leq, \Rightarrow^{\mathcal{A}}; D^{\mathcal{A}})$, an *implicational Routley-Meyer-style frame* (briefly, R-M$_{\Rightarrow}$ frame) for $\mathcal{A}$ is meant a structure $\mathcal{U} = (U, \leq, R_{\Rightarrow^{\mathcal{A}}}; D)$, where $(U, \leq)$ is a partially ordered set and $R_{\Rightarrow^{\mathcal{A}}} \subseteq U^3$ satisfies the postulate below: $(p_{\leq})$ $a \leq b$ iff there is $c \in D$ such that $R_{\Rightarrow^{\mathcal{A}}}(a, b; c)$, briefly $R_{\Rightarrow^{\mathcal{A}}}(a, b; D)$.

2. (L-R-M$_{\Rightarrow}$ frame) As in 1, for L an implicational tonoid logic (an implicational (residuated, dual residuated) partial Gaggle logic resp.), we can define L-R-M$_{\Rightarrow}$ frame.

Theorem 8 (Strong completeness)**.** L *is strongly complete w.r.t. the class of all* L*-R-M frames.*

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